PMATH 450 Notes

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1 Week 1

1.1 Borel Sets

Goals of 450/650:

1. Develop a theory of integration for functions $f : A \to \mathbb{R}, A \subset \mathbb{R}$, which is
   (a) More flexible
   (b) More rich
   (c) Still extends Riemann integration
2. Introduce Harmonic Analysis

General Outline (First Half)

1. Which sets should we integrate over? $\to$ Measurable Sets
2. Which functions should we try to integrate? $\to$ Measurable Functions

Definition:
Let X be a set. We call $\mathcal{A} \subseteq \mathcal{P}(X)$ a σ-algebra of subsets of X if:

1. $\emptyset \in \mathcal{A}$
2. $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$
3. $A_1, A_2, A_3, \dots \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

Remark: $\mathcal{A} \subseteq \mathcal{P}(x)$ is a σ-algebra.

1. $X \in \mathcal{A}$
   Proof:
   $$X \setminus \emptyset = X \in \mathcal{A}$$
2. $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$
   Proof:
   $$A \cup B = A \cup B \cup \emptyset \cup \emptyset \cup \cdots \in \mathcal{A}$$

3. $A_1, A_2, \dots \in \mathcal{A} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$

$$\bigcap_{i=1}^{\infty} A_i = X \setminus \left(\bigcup_{i=1}^{\infty} (X \setminus A_i) \right)$$

4. $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

Examples:

1. Smallest σ-algebra: $\{\emptyset, X\}$
2. Trivial σ-algebra: $\mathcal{P}(x)$
3. $\mathcal{A} = \{A \subseteq \mathcal{R} : A \text{ is open}\}$ is not a σ-algebra.

   Proof:

   Let $A = (0, 1) \in \mathcal{A}$

$$\mathbb{R} \setminus A = (-\infty, 0] \cup [1, \infty) \notin \mathcal{A}$$

4. $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ open or closed}\}$ is not a σ-algebra.

   Proof:

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \notin \mathcal{A}$$

Proposition:
X is a set, $\mathcal{C} \subseteq \mathcal{P}(x)$, then

$$\mathcal{A} := \bigcap \{\mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra}, \mathcal{C} \subseteq \mathcal{B}\}$$

is a σ-algebra.
It is the smallest σ-algebra containing $\mathcal{C}$.
Proof:
Piazza
Definition:
Let $\mathcal{C} = \{A \subseteq \mathbb{R} : A \text{ open}\}$
$\mathcal{A} = \bigcap \{B : \mathcal{C} \subseteq B, \mathcal{B} \text{ is a } \sigma\text{-algebra}\}$
is a σ-algebra. $\mathcal{A}$ is called the Borel σ-algebra.
The elements of $\mathcal{A}$ are called the Borel sets.
Remark:

1. Open Sets $\Rightarrow$ Borel Sets
2. Closed Sets $\Rightarrow$ Borel Sets
3. $\{x_1, x_2, \dots\} = \bigcup_{i=1}^{\infty} \{x_i\}$

   Countable Sets are always Borel sets.

   In particular, $\mathbb{Q}$ is a Borel Set.
4. $[a, b) = [a, b] \setminus \{b\} = [a, b] \cap (\mathbb{R} \setminus \{b\})$ is also a Borel set.

By points 3 and 4, we get a lot of Borel sets that are neither open nor closed.

1.2 Outer Measure

Idea

1. Given $A \subseteq \mathbb{R}$, how should we "measure" the "size" of A?
2. Some sets have "sizes" which "measure" more nicely than others. Which ones? Borel sets?

Goal
Define a function

$$m : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$$

(called a measure)
such that

1. $m((a, b)) = m([a, b]) = m((a, b]) = b - a$
2. $m(A \cup B) \leq m(A) + m(B)$
3. $A \cap B = \emptyset$, $m(A \cup B) = m(A) + m(B)$

Idea
$A \subseteq \mathbb{R}$, there exists bounded open intervals $I_i = (a_i, b_i)$ such that $A \subseteq \bigcup_{i=1}^{\infty} I_i$
We want:

$$\begin{aligned} m(A) &\leq \sum_{i=1}^{\infty} m(I_i) \\ &= \sum_{i=1}^{\infty} \ell(I_i) = \sum_{i=1}^{\infty}(b_i - a_i) \end{aligned}$$

Cover A by bounded, open intervals as finely as possible.
Definition:
We define (Lebesgue) outer measure by

$$m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$$

$$m^*(A) = \inf\{\sum_{i=1}^{\infty} \ell(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i, I_i \text{ are bounded and open interval}\}$$

Example: $\emptyset$
For any $\epsilon > 0$, $\emptyset \subseteq (0, \epsilon)$

$$\Rightarrow m^*(\emptyset) \leq \ell(0, \epsilon) = \epsilon$$

Since $m^*(\emptyset) \geq 0$, we have $m^*(\emptyset) = 0$
Example: $A = \{x_1, x_2, x_3, \dots\}$

$$A \subseteq \bigcup_{i=1}^{\infty}(x_i - \frac{\epsilon}{2^{i+1}}, x_i + \frac{\epsilon}{2^{i+1}})$$

$$\begin{aligned} m^*(A) &\leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \\ &= \frac{\epsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \\ &= \frac{\epsilon}{2}(\frac{1}{1 - \frac{1}{2}}) = \epsilon \end{aligned}$$

Since $\epsilon > 0$ was arbitrary,

$$m^*(A) = 0$$

Also, finite sets also have a measure of 0.
Goal
If I is an interval, then $m^*(I) = \ell(I)$.
Proposition: (Keywords: Subset, measure)
If $A \subseteq B$, then $m^*(A) \leq m^*(B)$ (Keywords: Monotone)
Why?
Let $X = \{\sum \ell(I_i) : A \subseteq \bigcup I_i\}$
Let $Y = \{\sum \ell(I_i) : B \subseteq \bigcup I_i\}$
We have $X \supseteq Y$
Then, we have $\inf X = m^*(A) \leq \inf Y = m^*(B)$.
Lemma
If $a, b \in \mathbb{R}$, with $a \leq b$, then

$$m^*([a,b]) = b - a$$

Proof
Let $\epsilon > 0$ be given. Since $[a,b] \subseteq (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$, we see that $m^*([a,b]) \leq b - a + \epsilon$.
Since $\epsilon > 0$ was arbitrary,

$$m^*([a,b]) \leq b - a$$

Let $I_i (i \in \mathbb{N})$ be bounded, open intervals such that $[a,b] \subseteq \bigcup_{i=1}^{\infty} I_i$.
Since $[a,b]$ is compact, there exists $n \in \mathbb{N}$ such that

$$[a,b] \subseteq \bigcup_{i=1}^{n} I_i$$

Therefore,

$$b - a \leq \sum_{i=1}^{n} \ell(I_i) \leq \sum_{i=1}^{\infty} \ell(I_i)$$

and so

$$m^*([a,b]) \geq b - a$$

Thus, $m^*([a,b]) = b - a$
Proposition:
If I is an interval, then $m^*(I) = \ell(I)$.
Proof:

1. Suppose I is bounded with endpoints $a \leq b$.

   Let $\epsilon > 0$

$$I \subseteq [a,b] \Rightarrow m^*(I) \leq b - a$$
$$[a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}] \subseteq I \Rightarrow b - a - \epsilon \leq m^*(I)$$
$$\Rightarrow b - a \leq m^*(I)$$

2. Suppose I is unbounded.

$$\forall n \in \mathbb{N}, \exists I_n \Rightarrow I_n \subseteq I, \ell(I_n) = n$$
$$\Rightarrow m^*(I) \geq m^*(I_n) = n$$
$$\Rightarrow m^*(I) = \infty = \ell(I)$$

1.3 Properties

Basic Properties of Outer Measure

Outer measure is

1. Translation Invariant
2. Countably Subadditive

Notation $x \in \mathbb{R}, A \subseteq \mathbb{R}$

$$x + A = \{x + a : a \in A\}$$

Proposition [Translation Invariant]

$$m^*(x + A) = m^*(A)$$

Why?

$$\begin{aligned}
m^*(x + A) &= \inf \left\{ \sum \ell(I_i) : x + A \subseteq \bigcup_{i=1}^{\infty} I_i \right\} \\
&= \inf \left\{ \sum \ell(I_i) : A \subseteq \bigcup_{i=1}^{\infty} (-x + I_i) \right\} \\
&= \inf \left\{ \sum \ell(-x + I_i) : A \subseteq \bigcup_{i=1}^{\infty} (-x + I_i) \right\} \\
&= \inf \left\{ \sum \ell(J_i) : A \subseteq \bigcup_{i=1}^{\infty} J_i \right\} \\
&= m^*(A)
\end{aligned}$$

Proposition: [Countable Subadditivity]
If $A_i \subseteq \mathbb{R}(i \in \mathbb{N})$, then $m^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} m^*(A_i)$
Proof
We may assume that each $m^*(A_i) < \infty$.
Let $\epsilon > 0$ be given and let's fix $i \in \mathbb{N}$.
There exists open bounded intervals $I_{i,j}$ such that $A_i \subseteq \bigcup_{i=1}^{\infty} I_{i,j}$ and

$$\sum_{j=1}^{\infty} \ell(I_{i,j}) \leq m^*(A_i) + \frac{\epsilon}{2^i}$$

We see that

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j} I_{i,j}$$

and so

$$\begin{aligned}
m^*(\bigcup_{i=1}^{\infty} A_i) &\leq \sum_{i,j} \ell(I_{i,j}) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \ell(I_{i,j}) \\
&\leq \sum_{i=1}^{\infty} \left(m^*(A_i) + \frac{\epsilon}{2^i} \right) \\
&= \sum_{i=1}^{\infty} (m^*(A_i)) + \epsilon
\end{aligned}$$

Corollary [Finite Subadditivity]
If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R})$, then $m^*(A_1 \cup \cdots \cup A_n) \leq m^*(A_1) + \cdots + m^*(A_n)$
Why?

$$A_1 \cup \cdots \cup A_n = A_1 \cup \cdots \cup A_n \cup \emptyset \cup \emptyset \cup \cdots$$

Problem
There exists $A, B \subseteq \mathbb{R}$ such that $A \cap B = \emptyset$ and $m^*(A \cup B) < m^*(A) + m^*(B)$
i.e., outer measure is not finitely additive .
Solution:
Restrict the domain of m^* to only include sets which measure "nicely".

2 Week 2

2.1 Measurable Sets

Goal
Restrict the domain of m^* to only include sets such that whenever $A \cap B = \emptyset$

$$m^*(A \cup B) = m^*(A) + m^*(B)$$

Definition:
We say $A \subseteq \mathbb{R}$ is **measurable** if $\forall X \subseteq \mathbb{R}$

$$m^*(X) = m^*(X \cap A) + m^*(X \setminus A)$$

Remark
Always,

$$m^*(X) \leq m^*(X \cap A) + m^*(X \setminus A)$$
$$X = (X \cap A) \cup (X \setminus A)$$

Remark
If $A \subseteq \mathbb{R}$ is measurable and $B \subseteq \mathbb{R}$ with $A \cap B = \emptyset$, then

$$\begin{aligned} m^*(A \cup B) &= m^*(X \cap A) + m^*(X \setminus A) \\ &= m^*(A) + m^*(B) \end{aligned}$$

Goal:
Show a lot of sets are measurable.
Prop:
If $m^*(A) = 0$, then A is measurable.
Proof
Let $X \subseteq \mathbb{R}$, since $X \cap A \subseteq A$
We have

$$0 \leq m^*(X \cap A) \leq m^*(A) = 0$$

and so $m^*(X \cap A) = 0$.

$$\begin{aligned} & m^*(X \cap A) + m^*(X \setminus A) \\ =& m^*(X \setminus A) \\ \leq& m^*(X) \end{aligned}$$

Proposition: $A_1, A_2, \dots, A_n$ measurable, then $\bigcup_{i=1}^n A_i$ is measurable.
Proof
It suffices to prove the result when $n = 2$. Let $A, B \subseteq \mathbb{R}$ be measurable.
Let $X \subseteq \mathbb{R}$.
Then,

$$\begin{aligned} m^*(X) &= m^*(X \cap A) + m^*(X \setminus A) \\ &= m^*(X \cap A) + m^*((X \setminus A) \cap B) + m^*((X \setminus A) \setminus B) \\ &= m^*(X \cap A) + m^*((X \setminus A) \cap B) + m^*(X \setminus (A \cup B)) \\ &\geq m^*((X \cap A) \cup ((X \setminus A) \cap B)) + m^*(X \setminus (A \cup B)) \\ &= m^*(X \cap (A \cup B)) + m^*(X \setminus (A \cup B)) \end{aligned}$$

Proposition: Let $A_1, A_2, \dots, A_n$ measurable, $A_i \cap A_j = \emptyset, i \neq j$.
Let $A = A_1 \cup \dots \cup A_n$.
If $X \subseteq \mathbb{R}$, then

$$m^*(X \cap A) = \sum_{i=1}^n m^*(X \cap A_i)$$

Proof:
When $n = 2$,
Let $A, B \subseteq \mathbb{R}$ be measurable with $A \cap B = \emptyset$. Let $X \subseteq \mathbb{R}$.
Then,

$$\begin{aligned} & m^*(X \cap (A \cup B)) \\ =& m^*((X \cap (A \cup B)) \cap A) + m^*((X \cap (A \cup B)) \setminus A) \\ =& m^*(X \cap A) + m^*(X \cap B) \end{aligned}$$

Corollary [Finite additivity]
$A_1, \dots, A_n$ measurable, $A_i \cap A_j = \emptyset$.
Then,

$$m^*(A_1 \cup \dots \cup A_n) = \sum_{i=1}^n m^*(A_i)$$

Proof

$$X = \mathbb{R}$$

2.2 Countable Additivity

Lemma: $A_i \subseteq \mathbb{R}$ means $(i \in \mathbb{N})$. If $A_i \cap A_j = \emptyset$ for $i \neq j$, then $A := \bigcup_{i=1}^\infty A_i$ is measurable.
Why?

$$B_n := A_1 \cup A_2 \cup \dots \cup A_n$$
$$X \subseteq \mathbb{R}$$

$$\begin{aligned} m^*(X) &= m^*(X \cap B_n) + m^*(X \setminus B_n) \\ &\geq m^*(X \cap B_n) + m^*(X \setminus A) \\ &\overset{\text{prop}}{=} \sum_{i=1}^n m^*(X \cap A_i) + m^*(X \setminus A) \end{aligned}$$

Taking $n \to \infty$

$$m^*(X) \geq \sum_{i=1}^{\infty} m^*(X \cap A_i) + m^*(X \setminus A$$
$$\geq m^*\left(\bigcup_{i=1}^{\infty}(X \cap A_i)\right) + m^*(X \setminus A)$$
$$= m^*(X \cap A) + m^*(X \setminus A)$$

Proposition $A \subseteq \mathbb{R}$ is measurable, then $\mathbb{R} \setminus A$ is measurable.
Why?

$$X \subseteq \mathbb{R}$$

$$m^*(X \cap (\mathbb{R} \setminus A)) + m^*(X \setminus (\mathbb{R} \setminus A))$$
$$= m^*(X \setminus A) + m^*(X \cap A)$$
$$= m^*(X)$$

Proposition: $A_i \subseteq \mathbb{R}$ measurable ($i \in \mathbb{N}$), then $A = \bigcup_{i=1}^{\infty} A_i$ is measurable.
Why? $B_1 = A_1$

$$B_n = A_n \setminus (A_1 \cup A_2 \cup \cdots \cup A_{n-1}), n \geq 2$$
$$B_n = A_n \cap (\mathbb{R} \setminus (A_1 \cup \cdots \cup A_{n-1}))$$

Therefore, B_n is a measurable set.
For $i \neq j$, $B_i \cap B_j = \emptyset$.
Also, $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.
Corollary
The collection $\mathcal{L}$ of (Lebesgue) measurable sets is a σ-algebra of sets in $\mathbb{R}$.
Proposition [Countable Additivity]
$A_i \subseteq \mathbb{R}$ means $i \in \mathbb{N}$ if $A_i \cap A_j = \emptyset$ for $i \neq j$.
Then:

$$m^*(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m^*(A_i)$$

Why?

$$m^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} m^*(A_i)$$

$$m^*(\bigcup_{i=1}^{\infty} A_i) \geq m^*(\bigcup_{i=1}^{n} A_i)$$
$$= \sum_{i=1}^{n} m^*(A_i)$$

Take $n \to \infty$.

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} m^*(A_i)$$

2.3 Borel Implies Measurable

Goal: Show Borel sets are measurable.

Proposition:

If $a \in \mathbb{R}$ then (a, ∞) is measurable.

Proof:

Let $X \subseteq \mathbb{R}$. We want to show that

$$m^*(X \cap (a, \infty) + m^*(X \setminus (a, \infty)) \leq m^*(X)$$

Case 1: $a \notin X$

We show: $m^*(X \cap (a, \infty)) + m^*(X \cap (-\infty, a)) \leq m^*(X)$.

Let the first outer measure be X_1, the second one be X_2.

Let (I_i) be a sequence of bounded, open intervals such that $X \subseteq \bigcup I_i$.

Define $I_i' = I_i \cap (a, \infty)$ and $I_i'' = I_i \cap (-\infty, a)$

Note that

$$X_1 \subseteq \bigcup I_i', X_2 \subseteq \bigcup I_i''$$

and so

$$m^*(X_1) \leq \sum \ell(I_i')$$

and

$$m^*(X_2) \leq \sum \ell(I_i'')$$

We then see that

$$\begin{aligned} &m^*(X_1) + m^*(X_2) \\ &\leq \sum \ell(I_i') + \sum \ell(I_i'') \\ &= \sum [\ell(I_i') + \ell(I_i)''] \\ &= \sum \ell(I_i) \end{aligned}$$

By the definition of inf,

$$m^*(X_1) + m^*(X_2) \leq m^*(X)$$

Case 2: $a \in X$

Piazza

Hint: $X' = X \setminus \{a\}$.

Theorem

Every Borel set is measurable.

Why?

(a, ∞) is measurable.

$\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) = [a, \infty)$ is also measurable.

$\mathbb{R} \setminus [a, \infty) = (-\infty, a)$ is measurable.

$(a, b) = (a, \infty) \cap (-\infty, b)$ is measurable.

Every open set in $\mathbb{R}$ is measurable.

$$\mathcal{B} \subseteq \mathcal{L}$$

Definition

We call $m : \mathcal{L} \to [0, \infty) \cup \{\infty\}$ given by

$$m(A) = m^*(A)$$

Lebesgue measure

Piazza

Prove that $A \subseteq \mathbb{R}$ is measurable, then $x + A$ is measurable for any $x \in \mathbb{R}$.

2.4 Basic Properties of Lebesgue Measure

Prop [Excision Property]

$A \subseteq B$, A measurable, $m(A) < \infty$, then $m^*(B \setminus A) = m^*(B) - m(A)$.

Why?

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \setminus A) \\ &= m(A) + m^*(B \setminus A). \end{aligned}$$

Theorem [Continuity of Measure]

1. $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ measurable
$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} m(A_n)$$
2. $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ measurable
$$m(B_1) < \infty$$
$$m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$$

Proof:

1. Since $m(A_k) \le m(\bigcup A_i)$ for all $k \in \mathbb{N}$, we have
$$\lim_{n\to\infty} m(A_n) \le m\left(\bigcup A_i\right)$$
If there exists $k \in \mathbb{N}$ such that $m(A_k) = \infty$, then $\lim_{n\to\infty} m(A_n) = \infty$ and we are done.
Thus, we may assume that each $m(A_k) < \infty$.
For each $k \in \mathbb{N}$, let $D_k = A_k \setminus A_{k-1}$, $A_0 = \emptyset$.
Note:
   - The D_k's are measurable
   - The D_k's are pairwise disjoint
   - $\bigcup D_i = \bigcup A_i$.

   Thus,
$$\begin{aligned} & m\left(\bigcup A_i\right) \\ =& m\left(\bigcup D_i\right) \\ =& \sum_{i=1}^{\infty} m(D_i) \qquad \text{(Countable additivity)} \\ =& \sum_{i=1}^{\infty} (m(A_i) - m(A_{i-1})) \qquad \text{(Excision Property)} \\ =& \lim_{n\to\infty} \sum_{i=1}^{n} (m(A_i) - m(A_{i-1})) \\ =& \lim_{n\to\infty} m(A_n) - m(A_0) \\ =& \lim_{n\to\infty} m(A_n) \end{aligned}$$

2. For $k \in \mathbb{N}$, define

$$D_k = B_1 \setminus B_k$$

Note:

- D_k's measurable
- $D_1 \subseteq D_2 \subseteq D_3 \subseteq \dots$

By (1), $m\left(\bigcup D_i\right) = \lim_{n\to\infty} m(D_n)$.

We see that

$$\begin{aligned}\bigcup D_i &= \bigcup_{i=1}^{\infty}(B_1 \setminus B_i) \\ &= B_1 \setminus \left(\bigcap_{i=1}^{\infty} B_i\right),\end{aligned}$$

and so

$$\begin{aligned}&\lim_{n\to\infty} m(D_n) \\ =&m(\bigcup D_i) \\ =&m\left(B_1 \setminus \left(\bigcap B_i\right)\right) \\ =&m(B_1) - m(\bigcap B_i).\end{aligned}$$

However,

$$\begin{aligned}\lim_{n\to\infty} m(D_n) &= \lim_{n\to\infty} m(B_1) - m(B_n) \\ &= m(B_1) - \lim_{n\to\infty} m(B_n)\end{aligned}$$

Example:
$B_i = (i, \infty)$, $m(\bigcap B_i) = m(\emptyset) = 0$

$$\lim_{n\to\infty} m(B_n) = \infty$$

3 Week 3

3.1 Non-measurable Set

A non-measurable set.

Goals of the week:

1. Construct an example of a non-measurable set.

2. Construct an element in $\mathcal{L} \setminus \mathcal{B}$.

Lemma
$A \subseteq \mathbb{R}$ bounded, measurable, $\Lambda \subseteq \mathbb{R}$ bounded, countably infinite.
If $\lambda + A, \lambda \in \Lambda$ are pairwise disjoint, then $m(A) = 0$.
Why?

$$\bigcup_{\lambda}(\lambda+A) \text{ bounded, measurable set}$$

$$m\left(\bigcup_{\lambda}(\lambda+A)\right)<\infty$$

$$\begin{aligned} m\left(\bigcup_{\lambda}(\lambda+A)\right) &= \sum_{\lambda} m(\lambda+A) \\ &= \sum_{\lambda} m(A)<\infty \end{aligned}$$

Hence, $m(A)=0$.
Construction
Start with $\emptyset \neq A \subseteq \mathbb{R}$. Consider $a \sim b \Leftrightarrow a-b \in \mathbb{Q}$.
Then [Piazza] $\sim$ is an equivalence relation.
Let C_A denote a single choice of equivalence class representatives for A relative to $\sim$.
Remark
The set $\lambda+C_A, \lambda \in \mathbb{Q}$, are pairwise disjoint.

$$\begin{aligned} & x \in (\lambda_1+C_A) \cap (\lambda_2+C_A) \\ \Rightarrow & x=\lambda_1+a=\lambda_2+b, a, b \in C_A \\ \Rightarrow & a-b=\lambda_2-\lambda_1 \in \mathbb{Q} \\ \Rightarrow & a \sim b \Rightarrow a=b \\ \Rightarrow & \lambda_1=\lambda_2 \end{aligned}$$

Theorem [Vitali]
Every set $A \subseteq \mathbb{R}$ with $m^*(A)>0$ contains a non-measurable subset.
Proof:
By Quiz 1, we may assume A is bounded. Say $A \subseteq [-N, N]$, for some $N \in \mathbb{N}$.
Claim: C_A is non-measurable.
Assume C_A is measurable.
Let $\Lambda \subseteq \mathbb{Q}$ be bounded infinite.
By the lemma and remark,

$$m(C_A)=0$$

Let $a \in A$. Then, $a \sim b$ for some $b \in C_A$. In particular,

$$a-b=\lambda \in \mathbb{Q}$$

Moreover, $\lambda \in [-2N, 2N]$.
Taking $\Lambda_0=\mathbb{Q} \cap [-2N, 2N]$. We have that

$$A \subseteq \bigcup_{\lambda \in \Lambda_0}(\lambda+C_A)$$

$\lambda+C_A$ has measure 0.
Contradiction.
Corollary
There exists $A, B \subseteq \mathbb{R}$ such that

1. $A \cap B = \emptyset$,

2. $m^*(A \cup B) < m^*(A) + m^*(B)$

Why?
Let C be unmeasurable set.

$$\exists X \subseteq \mathbb{R}, m^*(X) < m^*(X \cap C) + m^*(X \setminus C)$$

Outer measurable is not finitely additive.

3.2 Cantor-Lebesgue Function

Recall: Cantor Set.

$$I = [0, 1]$$
$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$
$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

etc

$$C = \bigcap_{k=1}^{\infty} C_k$$

- Uncountable
- Closed

Proposition
The Cantor Set is Borel and has measure 0.
Why?
Closed $\Rightarrow$ Borel

$$C = \bigcap_{k=1}^{\infty} C_k$$

C_k's measurable, $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$

$$m(C_1) < \infty$$

.

By the Continuity of Measure,

$$m(c) = \lim_{k\to\infty} m(C_k)$$
$$= \lim_{k\to\infty} \frac{2^k}{3^k} = 0$$

Construction: C-L function

1. For $k \in \mathbb{N}$, $\mathcal{U}_k$ = union of open intervals deleted in the process of constructing $C_1, C_2, \dots, C_k$.
   i.e., $\mathcal{U}_k = [0, 1] \setminus C_k$

2. $\mathcal{U} = \bigcup_{k=1}^{\infty} \mathcal{U}_k$
   i.e., $\mathcal{U} = [0, 1] \setminus C$

3. Say $\mathcal{U}_k = I_{k,1} \cup I_{k,2} \cup \cdots \cup I_{k,2^k-1}$ (in order)
   Define:
$$\varphi : \mathcal{U}_k \to [0,1]$$
   by
$$\varphi\big|_{I_{k,i}} = \frac{i}{2^k}$$
   Example:
$$\mathcal{U}_1 = \left(\frac{1}{3}, \frac{2}{3}\right) \mapsto \frac{1}{2}$$
$$\begin{aligned} \mathcal{U}_2 = \quad & \left(\frac{1}{9}, \frac{2}{9}\right) \cup & \left(\frac{1}{3}, \frac{2}{3}\right) \cup & \quad \left(\frac{7}{9}, \frac{8}{9}\right) \\ & \mapsto \frac{1}{4} & \mapsto \frac{2}{4} & \quad \mapsto \frac{3}{4} \\ & & = \frac{1}{2} & \end{aligned}$$
   etc.

4. Define
$$\varphi : [0,1] \to [0,1]$$
   by: For $0 \neq x \in C$, $\varphi(x) = \sup\{\varphi(t) : t \in \mathcal{U} \cap [0,x)\}$. and $\varphi(0) = 0$.
   This is the Cantor-Lebesgue function.
   Things to know about φ:
   (a) φ is increasing. [Piazza]
   (b) φ is continuous.
      - φ is continuous on $\mathcal{U}$.
      - $x \in C, x \neq 0, 1$
        For large k, there exists $a_k \in I_{k,i}, b_k \in I_{k,i+1}$ such that
$$a_k < x < b_k$$
        But,
$$\varphi(b_k) - \varphi(a_k) = \frac{i+1}{2^k} - \frac{i}{2^k} = \frac{1}{2^k} \to 0$$
        No jump!
      - $x \in \{0, 1\}$
   (c) $\varphi : \mathcal{U} \to [0,1]$ is differentiable and $\varphi' = 0$
   (d) φ is onto.
$$\varphi(0) = 0, \varphi(1) = 1$$
   By IVT

3.3 Non-Borel Sets

Let φ be the C-L function, Consider $\psi : [0,1] \to [0,2]$ defined by

$$\psi(x) = x + \varphi(x)$$

1. ψ is **strictly** increasing.
2. ψ is continuous.
3. ψ is onto.

$\Rightarrow \psi$ is invertible

Properties

1. $\psi(C)$ is measurable and has positive measure.
2. ψ maps a particular (measurable) subset of C to a non-measurable set.

Proof:

1. By A1, ψ^{-1} is continuous.
   $\therefore \psi(C) = \left(\psi^{-1}\right)^{-1}(C)$ is closed.
   $\Rightarrow \psi(C)$ is measurable.
   Note that
   $$[0,1] = C \sqcup \mathcal{U}$$
   $$\Rightarrow [0,2] = \psi(C) \sqcup \psi(\mathcal{U})$$
   $$\Rightarrow 2 = m(\psi(C)) + m(\psi(\mathcal{U}))$$
   It suffices to show that
   $$m(\psi(\mathcal{U})) = 1$$
   Say $\mathcal{U} = \bigsqcup_{i=1}^{\infty} I_i$, where the I_i's are disjoint open intervals.
   Then,
   $$\psi(\mathcal{U}) = \bigsqcup_{i=1}^{\infty} \psi(I_i)$$
   $$m(\psi(\mathcal{U})) = \sum m(\psi(I_i))$$
   Note that $\forall i \in \mathbb{N}$, $\exists r \in \mathbb{R}$, such that $\phi(x) = r$ for all $x \in I_i$.
   In particular, $\psi(x) = x + r$ for all $x \in I_i$ and so
   $$\psi(I_i) = r + I_i$$
   $\therefore m\left(\psi(\mathcal{U})\right) = \sum m(I_i) = m\left(\bigsqcup I_i\right) = m(\mathcal{U})$
   Since $[0,1] = \mathcal{U} \sqcup C$, we have that
   $$1 = m(\mathcal{U}) + m(C) = m(\mathcal{U})$$
   Hence,
   $$m\left(\psi(\mathcal{U})\right) = m\left(\mathcal{U}\right) = 1 > 0$$

2. By Vitali, $\psi(C)$ contains a subset $A \subseteq \psi(C)$ which is non-measurable.
   Let $B = \psi^{-1}(A) \subseteq C$.
   then, $\psi(B) = A$ is non-measurable as required.

Theorem
The Cantor set contains an element of $\mathcal{L} \setminus \mathcal{B}$
Why?

$$B \subseteq C \Rightarrow B \text{ measurable}$$
$$\psi(B) \text{ non-measurable}$$

By assignment 1, if B is Borel, then $\psi(B)$ is Borel.
Therefore, B is NOT Borel.

4 Week 4

4.1 Measurable Functions

Question: Which functions are suitable for integration?
Definition:
For $A \subseteq \mathbb{R}$ measurable, we say $f : A \to \mathbb{R}$ is measurable iff for all open $\mathcal{U} \subseteq \mathbb{R}$, $f^{-1}(\mathcal{U})$ is measurable.
Proposition:
If $A \subseteq \mathbb{R}$ is measurable and $f : A \to \mathbb{R}$ is continuous, then f is measurable.
Why?
$\mathcal{U} \subseteq \mathbb{R}$ is open $f^{-1}(\mathcal{U})$ open $\Rightarrow$ measurable.
Proposition: (Characteristic Function)
$A \subseteq \mathbb{R}$ measurable

$$\chi_A : \mathbb{R} \to \mathbb{R}, \chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then χ_A is measurable.
Why?
$\mathcal{U} \subseteq \mathbb{R}$ open

$$\chi_A^{-1}(\mathcal{U}) = \mathbb{R}, A, \mathbb{R} \setminus A, \emptyset.$$

Proposition: $A \subseteq \mathbb{R}$ measurable, $f : A \to \mathbb{R}$, the following are equivalent:

1. f is measurable
2. $\forall a \in \mathbb{R}$, $f^{-1}(a, \infty)$ is measurable
3. $\forall a < b$, $f^{-1}(a, b)$ is measurable.

Proof
(1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3)
Let $b \in \mathbb{R}$ so that $f^{-1}(b, \infty)$ is measurable. Then,

$$\begin{aligned} \mathbb{R} \setminus f^{-1}(b, \infty) &= f^{-1}(\mathbb{R} \setminus (b, \infty)) \\ &= f^{-1}\left((-\infty, b]\right) \end{aligned}$$

is measurable as well.
We see that

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$$

and so

$$f^{-1}(-\infty, b) = \bigcup_{n=1}^{\infty} f^{-1}(-\infty, b - \frac{1}{n}]$$

Each of the preimage $(-\infty, b - \frac{1}{n}]$ is measurable.
Finally, for $a < b$.

$$(a, b) = (a, \infty) \cap (-\infty, b)$$

$$\begin{aligned}\Rightarrow & f^{-1}(a, b) \\ & = f^{-1}(a, \infty) \cap f^{-1}(-\infty, b)\end{aligned}$$

is measurable.
(3) $\Rightarrow$ (1) is trivial.

4.2 Properties

Properties of measurable functions

Proposition:
$A \subseteq \mathbb{R}$ measurable, $f, g : A \to \mathbb{R}$ measurable.

1. For all $a, b \in \mathbb{R}$

$$af + bg$$

is measurable.

2. The function fg is measurable.

Proof

1. Let $a \in \mathbb{R}$. For $\alpha \in \mathbb{R}$,

$$(af)^{-1}(\alpha, \infty) = \{x \in A : af(x) > \alpha\}$$

(a) $a > 0$

$$\begin{aligned}(af)^{-1}(\alpha, \infty) &= \left\{x \in A : f(x) > \frac{\alpha}{a}\right\} \\ &= f^{-1}\left(\frac{\alpha}{a}, \infty\right) \\ &\to \text{measurable}\end{aligned}$$

(b) $a < 0$

$$\begin{aligned}(af)^{-1}(\alpha, \infty) &= f^{-1}\left(-\infty, \frac{\alpha}{a}\right) \\ &\to \text{measurable}\end{aligned}$$

(c) $a = 0$ af continuous $\Rightarrow$ measurable.

We now show that $f+g$ is measurable.

For $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&(f+g)^{-1}(\alpha,\infty)\\
=&\{x \in A : f(x)+g(x) > \alpha\}\\
=&\{x \in A : f(x) > \alpha - g(x)\}\\
=&\{x \in A : \exists q \in \mathbb{Q}, f(x) > q > \alpha - g(x)\}\\
=&\bigcup_{q\in\mathbb{Q}} (\{x \in A : f(x) > q\} \cap \{x \in A : g(x) > \alpha - q\})\\
=&\bigcup_{q\in\mathbb{Q}} \left(f^{-1}(a,\infty) \cap g^{-1}\left(\alpha - q, \infty\right)\right)
\end{aligned}$$

is measurable.

Hence, $f+g$ is measurable.

2. By the quiz, $|f|$ is measurable.

   For $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&(f^2)^{-1}(\alpha,\infty)\\
=&\{x \in A : f(x)^2 > \alpha\}\\
=&\begin{cases} A & \alpha < 0\\ \{x \in A : |f|(x) > \sqrt{\alpha}\} & \alpha \geq 0\end{cases}\\
=&\begin{cases} A & \alpha < 0\\ |f|^{-1}\left(\sqrt{\alpha},\infty\right) & \alpha \geq 0\end{cases}
\end{aligned}$$

   is measurable.

   Thus, f^2 is measurable.

   Since

$$(f+g)^2 = f^2 + 2fg + g^2$$

   is measurable, we have that $2fg$ is measurable.

   By part (1), fg is measurable.

<u>Exercise</u>

$\psi : [0,1] \to \mathbb{R}, \psi(x) = x + \varphi(x)$(Cantor-Lebesgue function)

$\exists A \subseteq [0,1]$ such that A is measurable but $\psi(A)$ is not measurable.

Extend $\psi : \mathbb{R} \to \mathbb{R}$ continuously to a strictly increasing surjective function such that ψ^{-1} is continuous.

[Piazza: How?]

Consider $\chi_A \circ \psi^{-1}$

Then,

$$\begin{aligned}
&\left(\chi_A \circ \psi^{-1}\right)^{-1}\left(\frac{1}{2},\frac{3}{2}\right)\\
=&\psi\left(\chi_A^{-1}\left(\frac{1}{2},\frac{3}{2}\right)\right)\\
=&\psi(A) \text{ NOT measurable}
\end{aligned}$$

Therefore, $\chi_A \circ \psi^{-1}$ is not measurable.

Proposition:
$A \subseteq \mathbb{R}$ measurable.
If $g : A \to \mathbb{R}$ is measurable and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f \circ g$ is measurable.
Why?
$\mathcal{U} \subseteq \mathbb{R}$ open

$$\begin{aligned} &(f \circ g)^{-1}(\mathcal{U}) \\ =&g^{-1}\left(f^{-1}(\mathcal{U})\right) \end{aligned}$$

is measurable.

4.3 More Properties

Define
$A \subset \mathbb{R}$
We say a property $P(x), x \in A$ is true almost everywhere (ae) if

$$m(\{x \in A : P(x) \text{ false}\}) = 0$$

Proposition
$f : A \to \mathbb{R}$ measurable.
If $g : A \to \mathbb{R}$ is a function and $f = g$ almost everywhere, then g is measurable.
Why?

$$\begin{aligned} B = \{x \in A : f(x) \neq g(x)\} \\ m(B) = 0 \end{aligned}$$

Let $\alpha \in \mathbb{R}$.

$$\begin{aligned} g^{-1}(\alpha, \infty) &= \{x \in A : g(x) > \alpha\} \\ &= \{x \in A \setminus B : g(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= \{x \in A \setminus B : f(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= \left(f^{-1}(\alpha, \infty) \cap A \setminus B\right) \cup \{x \in B : g(x) > \alpha\} \end{aligned}$$

is measurable.
Proposition:
A measurable, $B \subseteq A$ measurable.
A function $f : A \to \mathbb{R}$ is measurable iff $f|_B$ and $f|_{A \setminus B}$ are measurable.
Proof:

- Forward direction:

  Suppose $f : A \to \mathbb{R}$ is measurable. Let $\alpha \in \mathbb{R}$. Then,

  $$\begin{aligned} (f|_B)^{-1}(\alpha, \infty) &= \{x \in B | f(x) > \alpha\} \\ &= f^{-1}(\alpha, \infty) \cap B \end{aligned}$$

  is measurable.

  Therefore, $f|_B$ is measurable.

  The proof for $f|_{A \setminus B}$ is identical.

- Reverse direction:

  Suppsoe $f|_B$ and $f|_{A\setminus B}$ are measurable. For $\alpha \in \mathbb{R}$ are measurable. For $\alpha \in \mathbb{R}$,

$$\begin{aligned} f^{-1}(\alpha, \infty) &= \{x \in A : f(x) > \alpha\} \\ &= \{x \in B : f(x) > \alpha\} \cup \{x \in A \setminus B : f(x) > \alpha\} \\ &= (f|_B)^{-1}(\alpha, \infty) \cup \left(f|_{A\setminus B}\right)^{-1}(\alpha, \infty) \end{aligned}$$

  is measurable and so f is a measurable function.

Proposition
(f_n) measurable, $A \to \mathbb{R}$.
If $f_n \to f$ pointwise almost everywhere, then f is measurable.
Proof:
Let $B = \{x \in A : f_n(x) \not\to f(x)\}$.
So that $m(B) = 0$.
For $\alpha \in \mathbb{R}$,

$$(f|_B)^{-1}(\alpha, \infty) = f^{-1}(\alpha, \infty) \cap B$$

is measurable.
A function whose domain has measure 0 is measurable.
It suffices to show that $f|_{A\setminus B}$ is measurable.
By replacing f by $f|_{A\setminus B}$, we may assume $f_n \to f$ pointwise.
Let $\alpha \in \mathbb{R}$. Since $f_n \to f$ pointwise, we see that for $x \in A$:

$$\begin{aligned} &f(x) > \alpha \\ \Leftrightarrow &\exists n, N \in \mathbb{N}, \forall i \geq N, f_i(x) > \alpha + \frac{1}{n} \end{aligned}$$

We then see that

$$\begin{aligned} &f^{-1}(\alpha, \infty) \\ =&\bigcup_{n\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{i=N}^{\infty} f_i^{-1}\left(\alpha + \frac{1}{n}, \infty\right) \end{aligned}$$

is measurable.
Therefore, f is measurable.

4.4 Simple Approximation

Definition:
A function $\varphi : A \to \mathbb{R}$ is called simple if

1. φ is measurable.
2. $\varphi(A)$ is finite.

Remark [Canonical Representation]
$\varphi : A \to \mathbb{R}$ simple. $\varphi(A) = \{c_1, c_2, \ldots, c_k\}$ distinct
$A_i = \varphi^{-1}(\{c_i\})$ measurable.

- $A = \bigsqcup_{i=1}^{k} A_i$.

- $\varphi = \sum_{i=1}^{k} c_i \chi_{A_i}$

Goal:
Show measurable functions can be approximated by simple functions.
Lemma: $f : A \to \mathbb{R}$ measurable and bounded.
For all $\epsilon > 0$, there exists simple $\varphi_\epsilon, \psi_\epsilon : A \to \mathbb{R}$ such that

1. $\varphi_\epsilon \leq f \leq \psi_\epsilon$ and
2. $0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon$.

Why?
$f(A) \subseteq [a, b]$, $\epsilon > 0$

$$a = y_0 < y_1 < y_2 < \cdots < y_n = b$$
$$y_{i+1} - y_i < \epsilon$$

$$I_k = [y_{k-1}, y_k), A_k = f^{-1}(I_k)$$

A_k is measurable.
$\varphi_\epsilon : A \to \mathbb{R}, \psi_\epsilon : A \to \mathbb{R}$.

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \chi_{A_k}$$
$$\psi_\epsilon = \sum_{k=1}^{n} y_k \chi_{A_k}$$

The two functions are both simple.
Let $x \in A$. Since $f(x) \in [a, b]$, there exists $k \in \{1, \dots, n\}$ such that $f(x) \in I_k$.
i.e., $y_{k-1} \leq f(x) < y_k$, $x \in A_k$.
Moreover,

$$\varphi_\epsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\epsilon(x)$$

and so:

$$\varphi_\epsilon \leq f < \psi_\epsilon.$$

For the same x,

$$0 \leq \psi_\epsilon(x) - \varphi_\epsilon(x) = y_k - y_{k-1} < \epsilon.$$

Theorem [Simple Approximation]
$A \subseteq \mathbb{R}$ measurable.
A function $f : A \to \mathbb{R}$ is measurable iff there is a sequence (φ_n) of simple functions on A such that

1. $\varphi_n \to f$ pointwise.
2. $\forall n, |\varphi_n| \leq |f|$

Proof:

- Backwards direction: Done.

- Forward direction:

  Suppose $f : A \to \mathbb{R}$ is measurable.

  1. $f \geq 0$:
     For each $n \in \mathbb{N}$, define:
     $$A_n = \{x \in A : f(x) \leq n\}$$
     so that A_n is measurable and $f|_{A_n}$ is measurable and bounded.
     By the lemma, there exists simple functions $(\varphi_n), (\psi_n)$ such that
     $$0 \leq \varphi_n \leq f \leq \psi_n$$
     on A_n and
     $$0 \leq \psi_n - \phi_n < \frac{1}{n}$$
     .
     Fix $n \in \mathbb{N}$.
     Extend $\phi_n : A \to \mathbb{R}$ by setting $\varphi_n(x) = n$ if $x \notin A_n$.
     Therefore, $0 \leq \varphi_n \leq f$.
     For each $n \in \mathbb{N}$,
     $$\varphi_n : A \to \mathbb{R}$$
     is simple.
     <u>Claim:</u> $\varphi_n \to f$ pointwise.
     Let $x \in A$ and let $N \in \mathbb{N}$ such that $f(x) \leq N$ (i.e., $x \in A_n$).
     For $n \geq N$, $x \in A_n$ and so
     $$0 \leq f(x) - \varphi_n(x) \leq \psi_n(x) - \varphi_n(x) < \frac{1}{n}.$$

  2. $f : A \to \mathbb{R}$ is measurable.
     We let $B = \{x \in A : f(x) \geq 0\}, C = \{x \in A : f(x) < 0\}$ be measurable.
     We define $g, h : A \to \mathbb{R}$:
     $$g = \chi_B f, h = -\chi_C f$$
     so that g, h are measurable and non-negative.
     By Case 1, there exists sequences $(\varphi_n), (\psi_n)$ of simple functions such that $\varphi_n \to g$ pointwise, $\psi_n \to h$ pointwise, $0 \leq \varphi_n \leq g, 0 \leq \psi_n \leq h$.
     Then,
     $$\varphi_n - \psi_n \to g - h = f$$
     pointwise.
     and
     $$\begin{aligned} |\varphi_n - \psi_n| &\leq |\varphi_n| + |\psi_n| \\ &= \varphi_n + \psi_n \\ &\leq g + h = |f|. \end{aligned}$$

5 Week 5

5.1 Littlewood 1

Littlewood's Principles

Up to certain finiteness conditions:

1. Measurable sets are "almost" finite, disjoint union of bounded open intervals.
2. Measurable functions are "almost" continuous.
3. Pointwise limit of measurable functions are "almost" uniform limits.

Theorem [Littlewood 1]
A measurable with finite measure, $m(A) < \infty$.
For all $\epsilon > 0$, there exists finitely many open bounded, disjoint intervals $I_1, I_2, \ldots, I_n$ such that:

$$m\left(A \triangle \mathcal{U}\right) < \epsilon,$$

where $\mathcal{U} = I_1 \cup I_2 \cup \cdots \cup I_n$.
Note: $m\left(A \triangle \mathcal{U}\right) = m\left(A \setminus \mathcal{U}\right) + m\left(\mathcal{U} \setminus A\right)$
Proof
Let $\epsilon > 0$ be given.
We may find an open set $\mathcal{U}$ such that $A \subseteq \mathcal{U}$ and

$$m\left(\mathcal{U} \setminus A\right) < \boxed{\epsilon/2}$$

By PMATH 351, there exists bounded, open, disjoint intervals $I_i (i \in \mathbb{N})$ such that:

$$\mathcal{U} = \bigsqcup_{i=1}^{\infty} I_i$$

Note that:

$$\sum_{i=1}^{\infty} \ell(I_i) = m(\mathcal{U}) < \infty$$

That tells us that this series converges.
In particular, there exists $N \in \mathbb{N}$ such that:

$$\sum_{i=N+1}^{\infty} \ell(I_i) < \boxed{\epsilon/2}$$

Take $V = I_1 \cup \cdots \cup I_N$.
We see that,

$$\begin{aligned} m\left(A \setminus V\right) &\leq m\left(\mathcal{U} \setminus V\right) \\ &= m\left(\bigcup_{N+1}^{\infty} I_i\right) \\ &= \sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{2}. \end{aligned}$$

And:

$$m\left(V \setminus A\right) \leq m\left(\mathcal{U} \setminus A\right) < \frac{\epsilon}{2}$$

Therefore, $m\left(A \triangle V\right) < \epsilon$

5.2 Littlewood 3

Goal: Prove that pointwise limits of measurable functions are almost uniform limits.

Lemma

A measurable, $m(A) < \infty$, (f_n) measurable, $A \to \mathbb{R}$.

Assume $f : A \to \mathbb{R}$ such that $f_n \to f$ pointwise.

For all $\alpha, \beta > 0$, there exists a measurable subset $B \subseteq A$ and $N \in \mathbb{N}$ such that

1. $|f_n(x) - f(x)| < \alpha$ for all $x \in B, n \geq N$.
2. $m(A \setminus B) < \beta$.

Proof:

Let $\alpha, \beta > 0$ be given.

For $n \in \mathbb{N}$, define

$$\begin{aligned} A_n &= \{x \in A : |f_k(x) - f(x)| < \alpha \text{ for all } k \geq n\} \\ &= \bigcap_{k=n}^{\infty} |f_k - f|^{-1} (-\infty, \alpha) \end{aligned}$$

Measurable.

Therefore, every A_n is measurable.

Since $f_n \to f$ pointwise,

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Since (A_n) is ascending, by the continuity of measure:

$$m(A) = \lim_{n \to \infty} m(A_n) < \infty.$$

We may find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$m(A) - m(A_n) < \beta.$$

Pick $B = A_N$.

Theorem [Littlewood 3, Egoroff's Theorem]

A measurable, $m^*(A) = m(A) < \infty$. (f_n) measurable, $A \to \mathbb{R}$, $f_n \to f$ pointwise.

For all $\epsilon > 0$, there exists a closed set $C \subseteq A$ such that:

1. $f_n \to f$ uniform on C.
2. $m(A \setminus C) < \epsilon$

Proof

Let $\epsilon > 0$ be given.

By the lemma, for every $n \in \mathbb{N}$, there exists a measurable set $A_n \subseteq A$ and $N(n) \in \mathbb{N}$ such that:

1. For all $x \in A_n$ and $x \geq N(n)$,
$$|f_k(x) - f(x)| < \frac{1}{n}$$
2. $m(A \setminus A_n) < \boxed{\text{Stuff}}$

Take $B = \bigcap_{n=1}^{\infty} A_n$ (measurable).
For $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, $k \geq N(n)$, and $x \in B$

$$|f_k(x) - f(x)| < \frac{1}{n} < \epsilon$$

Therefore, $f_n \to f$ uniformly on B.
Moreover,

$$\begin{aligned} m(A \setminus B) &= m\left(A \setminus \bigcap A_n\right) \\ &= m\left(\bigcup (A \setminus A_n)\right) \\ &\leq \sum m(A \setminus A_n) \\ &< \sum \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2} \end{aligned}$$

By A1, there exists a closed set C such that $C \subseteq B$ and $m(B \setminus C) < \frac{\epsilon}{2}$.

1. Since $C \subseteq B$, $f_k \to f$ uniformly on C
2. $m(A \setminus C) = m(A \setminus B) + m(B \setminus C) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Example: Warning
$f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$. $f_n \to 0$ pointwise.
[Piazza]
$f_n \not\to 0$ uniformly on any measurable set $B \subseteq \mathbb{R}$ such that $m(\mathbb{R} \setminus B) < 1$.
Need: $m(A) < \infty$.

5.3 Littlewood 2

Goal: Prove that measurable functions are "almost" continuous.
i.e. Littlewood's 2nd Principle / Lusin's Theorem
Lemma
$f : A \to \mathbb{R}$ simple
For all $\epsilon > 0$, there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed $C \subseteq A$ such that

1. $f = g$ on C.
2. $m(A \setminus C) < \epsilon$.

Why?
$f = \sum_{i=1}^{n} a_i \chi_{A_i}$: Canonical Representation.
$A_i = \{x \in A : f(x) = a_i\}$ measurable
A1 $\Rightarrow C_i \subseteq A_i$ closed such that:

$$m(A_i \setminus C_i) < \boxed{\frac{\epsilon}{n}}$$

$A = \bigsqcup_{i=1}^{n} A_i, C := \bigsqcup_{i=1}^{n} C_i$ closed.

1. For all $x \in C_i$, $f(x) = a_i$.

   A1 $\Rightarrow$ f is continuous on C.

   A1 $\Rightarrow$ We then extend $f|_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$.
2. $m(A \setminus C) = m\left(\bigsqcup_{i=1}^{n} (A_i \setminus C_i)\right) = \sum_{i=1}^{n} m(A_i \setminus C_i) < \epsilon$

Theorem [Littlewood 2, Lusin's Theorem]
$f : A \to \mathbb{R}$ measurable.
For all $\epsilon > 0$, there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed set $C \subseteq A$ such that:

1. $f = g$ on C and

2. $m(A \setminus C) < \epsilon$.

Proof
Let $\epsilon > 0$ be given.
Case 1: $m(A) < \infty$.
Let $f : A \to \mathbb{R}$ be measurable.
By the Simple Approximation Theorem, there exists (f_n) simple such that $f_n \to f$ pointwise.
By the Lemma, there exists continuous function $g_n : \mathbb{R} \to \mathbb{R}$ and closed sets $C_n \subseteq A$ such that

1. $f_n - g_n$ on C_n and

2. $m(A \setminus C_n) < \boxed{\text{Stuff}}$

By Egoroff, there exists a closed set $C_0 \subseteq A$ such that $f_n \to f$ uniformly on C_0 and $m(A \setminus C_0) < \boxed{\frac{\epsilon}{2}}$.
Let $C = \bigcap_{i=1}^{\infty} C_i$.

1. $g_n = f_n \to f$ uniformly on $C \subseteq C_0$

   Therefore, f is continuous on C.

   A1: We may extend $f|_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$.

2.

$$\begin{aligned} m(A \setminus C) &= m\left(A \setminus \bigcap_{i=0}^{\infty} C_i\right) \\ &= m\left(\bigcup_{i=0}^{\infty} (A \setminus C_0)\right) \\ &\leq \sum_{i=0}^{\infty} m(A \setminus C_i) \\ &= m(A \setminus C_0) + \sum_{i=1}^{\infty} m(A \setminus C_i) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Case 2: $m(A) = \infty$.
For $n \in \mathbb{N}$,

$$A_n := \{a \in A : |a| \in [n-1, n)\}$$

so that $A = \bigsqcup_{n=1}^{\infty} A_n$.
By Case 1, there exists continuous functions $g_n : \mathbb{R} \to \mathbb{R}$ and closed $C_n \subseteq A_n$ such that

1. $f = g_n$ on C_n

2. $m(A_n \setminus C_n) < \boxed{\frac{\epsilon}{2^n}}$

Consider $C = \bigsqcup_{n=1}^{\infty} C_n$.
[Piazza] C is closed.

1.

$$\begin{aligned} m(A \setminus C) &= m\left(\bigsqcup (A_n \setminus C_n)\right) \\ &= \sum m(A_n \setminus C_n) \\ &< \epsilon \end{aligned}$$

2. $g : C \to \mathbb{R}$:

   Let $x \in C$ so that $x \in C_n$ for exactly one $n \in \mathbb{N}$.

   Define $g(x) = g_n(x) = f(x)$.

   [Piazza] Then, g is continuous.

   A1 $\Rightarrow$ Extend g continuously to all of $\mathbb{R}$.

6 Week 6

6.1 Integration 1

1. Simple functions:
$$\varphi : A \to \mathbb{R}, m(A) < \infty$$
2. $f : A \to \mathbb{R}$ bounded measurable
$$m(A) < \infty, \varphi_\epsilon \leq f \leq \psi_\epsilon$$
3. $f : A \to \mathbb{R}$ measurable, $f \geq 0$.
$$\sup \left\{ \int_A h : h \in (2), 0 \leq h \leq f \right\}$$
4. $f : A \to \mathbb{R}$ measurable:
$$f^+ = \max\{f, 0\}$$
$$f^- = \max\{-f, 0\}$$

Step 1: $\varphi : A \to \mathbb{R}$ simple, $m(A) < \infty$.
Definition: $m(A) < \infty, \varphi : A \to \mathbb{R}$ simple.
Carnonical Representation:
$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$$
The (Lebesgue) integral of φ over A is:
$$\int_A \varphi = \sum_{i=1}^{n} a_i m(A_i)$$
Lemma $m(A) < \infty$ (A measurable).
If $B_1, B_2, \ldots, B_n \subseteq A$ are measurable and disjoint, and $\varphi : A \to \mathbb{R}$ is defined by
$$\varphi = \sum_{i=1}^{n} b_i \chi_{B_i}$$
then
$$\int_A \varphi = \sum_{i=1}^{n} b_i m(B_i)$$
Why?
For $n = 2$:
If $b_1 \neq b_2$, then $\varphi = b_1\chi_{B_1} + b_2\chi_{B_2}$ is the canonical representation.
If $b_1 = b_2$, then
$$\begin{aligned} b_1\chi_{B_1} + b_1\chi_{B_2} &= b_1 \left(\chi_{B_1} + \chi_{B_2}\right) \\ &= b_1\chi_{B_1 \cup B_2} \end{aligned}$$

Thus,

$$\begin{aligned}\int_A \varphi &= b_1 m\left(B_1 \sqcup B_2\right)\\ &= b_1\left(m(B_1)+m(B_2)\right)\\ &= b_1 m(B_1) + b_2 m(B_2)\end{aligned}$$

Proposition: $\varphi, \psi : A \to \mathbb{R}$ simple, $m(A) < \infty$.
For all $\alpha, \beta \in \mathbb{R}$.

$$\int_A (\alpha\varphi + \beta\psi) = \alpha \int_A \varphi + \beta \int_A \psi$$

Why?
Let

$$\begin{aligned}\varphi(A) &= \{a_1, a_2, \ldots, a_n\}\\ \psi(A) &= \{b_1, b_2, \ldots, b_m\}\end{aligned}$$

distinct.

$$\begin{aligned}C_{ij} &= \{x \in A : \varphi(x) = a_i, \psi(x) = b_j\}\\ &= \varphi^{-1}\left(\{a_i\}\right) \cap \psi^{-1}\left(\{b_j\}\right)\end{aligned}$$

measurable.

$$\alpha\varphi + \beta\psi = \sum_{i,j} \left(\alpha a_i + \beta b_j\right) \chi_{C_{ij}}$$

C_{ij} are pairwise disjoint.
By the lemma,

$$\begin{aligned}\int_A \alpha\varphi + \beta\psi &= \sum_{i,j} \left(\alpha a_i + \beta b_j\right) m\left(C_{ij}\right)\\ &= \sum_{i,j} \alpha a_i m\left(C_{ij}\right) + \sum_{i,j} \beta b_j m\left(C_{ij}\right)\\ &= \sum_i \alpha a_i \left(\sum_j m\left(C_{ij}\right)\right) + \sum_j \beta b_j \left(\sum_i m\left(C_{ij}\right)\right)\\ &= \sum_i \alpha a_i \left(m\left(\{x \in A : \varphi(x) = a_i\}\right)\right)\\ &\quad + \sum_j \beta b_j \left(m\left(\{x \in A : \psi(x) = b_j\}\right)\right)\\ &= \alpha \int_A \varphi + \beta \int_A \psi\end{aligned}$$

Proposition:
$\varphi, \psi : A \to \mathbb{R}$ simple, $m(A) < \infty$.
If $\varphi \leq \psi$, then

$$\int_A \varphi \leq \int_A \psi$$

Why?

$$\int_A \psi - \int_A \varphi$$
$$= \int_A (\psi - \varphi) \leq 0$$

6.2 Integration 2

Step 2:
$f : A \to \mathbb{R}$ bounded, measurable functions.

$$m(A) < \infty$$

Recall:
For all $\epsilon > 0$, the exist simple $\varphi_\epsilon \leq f \leq \psi_\epsilon$ such that $\psi_\epsilon - \varphi_\epsilon < \epsilon$.
Definition:
$f : A \to \mathbb{R}$ bounded measurable, $m(A) < \infty$.
Lower Lebesgue Integral:

$$\underline{\int_A} f = \sup \left\{ \int_A \varphi : \varphi \leq f \text{ simple} \right\}$$

Upper Lebesgue Integral:

$$\overline{\int_A} f = \inf \left\{ \int_A \psi : f \leq \psi \text{ simple} \right\}$$

Proposition: $m(A) < \infty, f : A \to \mathbb{R}$ bounded measurable.
Then:

$$\underline{\int_A} f = \overline{\int_A f}$$

Proof:
For all $n \in \mathbb{N}$, there exists simple functions:

$$\varphi_n, \psi_n : A \to \mathbb{R}$$

such that:

1. $\varphi_n \leq f \leq \psi_n$
2. $\psi_n - \varphi_n \leq \frac{1}{n}$

We see that,

$$0 \leq \overline{\int_A} f - \underline{\int_A} f$$
$$\leq \int_A \psi_n - \int_A \varphi_n$$
$$= \int_A (\psi_n - \varphi_n)$$
$$\leq \int_A \frac{1}{n} = \frac{1}{n} m(A) \to 0$$

Definition:
$m(A) < \infty, f : A \to \mathbb{R}$ bounded measurable.

We define the (Lebesgue) integral of f over A by:

$$\int_A f := \underline{\int_A} f = \overline{\int_A} f$$

<u>Proposition:</u> $f, g : A \to \mathbb{R}$ bounded measurable, $m(A) < \infty$.
For $\alpha, \beta \in \mathbb{R}$,

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

<u>Proof:</u>
[Piazza] Scalar multiplication.
$\varphi_1, \varphi_2, \psi_1, \psi_2$ all simple.

$$\varphi_1 \leq f \leq \psi_1, \varphi_2 \leq g \leq \psi_2$$

1.

$$\begin{aligned}\int_A f + g &= \overline{\int_A} f + g \\ &\leq \int_A (\psi_1 + \psi_2) \\ &= \int_A \psi_1 + \int_A \psi_2\end{aligned}$$

$$\begin{aligned}&\int_A f + g \\ &\leq \inf \left\{ \int_A \psi_1 + \int_A \psi_2 : f \leq \psi_1, g \leq \psi_2 \right\} \\ &= \inf \left\{ \int_A \psi_1 : f \leq \psi_1 \text{ simple} \right\} + \inf \left\{ \int_A \psi_2 : g \leq \psi_2 \text{ simple} \right\} \\ &= \int_A f + \int_A g\end{aligned}$$

2.

$$\begin{aligned}\int_A f + g &= \underline{\int_A} f + g \\ &\geq \int_A \varphi_1 + \varphi_2 \\ &= \int_A \varphi_1 + \int_A \varphi_2\end{aligned}$$

Similarly, by taking sup,

$$\int_A f + g \geq \int_A f + \int_A g$$

<u>Proposition:</u> $f, g : A \to \mathbb{R}$ bounded measurable, $m(A) < \infty$.
If $f \leq g$, then:

$$\int_A f \leq \int_A g$$

<u>Why?</u>

$$g - f \geq 0$$

$$\begin{aligned}
&\int_A (g - f) = \underline{\int_A} (g - f) \geq \int_A 0 = 0 \\
&\Rightarrow \int_A g - \int_A f \geq 0
\end{aligned}$$

6.3 BCT

Bounded Convergence Theorem

Proposition: $f : A \to \mathbb{R}$ bounded measurable, $B \subseteq A$ measurable, $m(A) < \infty$. Then

$$\int_B f = \int_A f\chi_B$$

Proof

1. $f = \chi_C$, $C \subseteq A$ measurable.

$$\begin{aligned}
\int_A \chi_C \chi_B &= \int_A \chi_{B\cap C} \\
&= m\left(B \cap C\right) \\
&= \int_B \chi_C\big|_B
\end{aligned}$$

2. f is simple, $f = \sum_{i=1}^n a_i \chi_{A_i}$.

   Thus,

$$\begin{aligned}
\int_A f\chi_B &= \sum a_i \int_A \chi_{A_i}\chi_B \\
&= \sum a_i \int_B \chi_{A_i} \\
&= \int_B \left(\sum a_i \chi_{A_i}\right) \\
&= \int_B f
\end{aligned}$$

3. $f : A \to \mathbb{R}$ be bounded, measurable functions.

   (a) $f \leq \psi$ simple

$$\int_A f\chi_B \leq \int_A \psi\chi_B = \int_B \psi$$

   By taking the inf over all such ψ, we have that

$$\int_A f\chi_B \leq \overline{\int_B} f = \int_B f$$

   Taking $\varphi \leq f$, φ is simple, we obtain:

$$\underline{\int_B} f = \int_B f \leq \int_A f\chi_B$$

Proposition: $f : A \to \mathbb{R}$ bounded, measurable, $m(A) < \infty$.
If $B, C \subseteq A$ are measurable and disjoint, then:

$$\int_{B\cup C} f = \int_B f + \int_C f$$

Why?

$$\begin{aligned}\int_{B\cup C} f &= \int_A f\chi_{B\cup C} \\ &= \int_A f\left(\chi_B + \chi_C\right) \\ &= \int_A f\chi_B + \int_A f\chi_C \\ &= \int_B f + \int_C f\end{aligned}$$

Proposition: $f : A \to \mathbb{R}$ bounded, measurable, $m(A) < \infty$.
Then

$$\left|\int_A f\right| \leq \int_A |f|$$

Why?

$$-|f| \leq f \leq |f|$$

$$-\int_A |f| \leq \int_A f \leq \int_A |f|$$

Proposition: (f_n) bounded, measurable, $A \to \mathbb{R}$, $m(A) < \infty$.
If $f_n \to f$ uniform, then $\lim_{n\to\infty} \int_A f_n = \int_A f$.
Proof.
Let $\epsilon > 0$ be given.
Let $N \in \mathbb{N}$ such that

$$|f_n - f| < \boxed{\tfrac{\epsilon}{m(A)+1}}$$

For $n \geq N$. Then, for $n \geq N$,

$$\begin{aligned}&\left|\int_A f_n - \int_A f\right| \\ =&\left|\int_A (f_n - f)\right| \\ \leq&\int_A |f_n - f| \\ \leq& m(A)\cdot\frac{\epsilon}{m(A)+1} < \epsilon\end{aligned}$$

Exercise:
$f_n : [0,1] \to \mathbb{R}$.

$$f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{n} \\ n & \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & x \geq \frac{2}{n} \end{cases}$$

$f_n \to 0$ pointwise.

$$\int_{[0,1]} f_n = 1$$

$$\int_{[0,1]} 0 = 0$$

Theorem [BCT]
(f_n) measurable, $A \to \mathbb{R}, m(A) < \infty$.
If there exists $M > 0$, such that $|f_n| \leq M$ for all n, and $f_n \to f$ pointwise, then:

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof:
Let $\epsilon > 0$ be given. By Egoroff's Theorem, there exists measurable $B \subseteq A$ and $N \in \mathbb{N}$ such that for $n \geq N$:

1. $|f_n - f| < \boxed{\frac{\epsilon}{2(m(B)+1)}}$ on B
2. $m(A \setminus B) < \boxed{\frac{\epsilon}{4M}}$

For $n \geq N$,

$$\begin{aligned}
\left| \int_A f_n - \int_A f \right| &\leq \int_A |f_n - f| \\
&= \int_B |f_n - f| + \int_{A\setminus B} |f_n - f| \\
&\leq \int_B |f_n - f| + \int_{A\setminus B} (|f_n| + |f|) \\
&\leq \int_B |f_n - f| + 2Mm(A \setminus B) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

6.4 Integration 3

$f : A \to \mathbb{R}$, $f \geq 0$ measurable.
Definition:
$f : A \to \mathbb{R}$ measurable.

1. We say f has finite support if
$$A_0 := \{x \in A : f(x) \neq 0\}$$
has finite measure.
2. We say f is a BF function if f is bounded and has finite support.
3. If $f : A \to \mathbb{R}$ is BF, then
$$\int_A f := \int_{A_0} f$$

Definition:
$f : A \to \mathbb{R}$ measurable, $f \geq 0$.

$$\int_A f := \sup \left\{ \int_A h : 0 \leq h \leq f \text{ BF} \right\}$$

Proposition: $f, g : A \to \mathbb{R}$ measurable, $f, g \geq 0$.

1. $\forall \alpha, \beta \in \mathbb{R}$:
$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

2. If $f \leq g$, then $\int_A f \leq \int_A g$.

3. If $B, C \subseteq A$ are measurable and $B \cap C = \emptyset$, then
$$\int_{B \cup C} f = \int_B f + \int_C f$$

<u>Proposition:</u> [Chebychev's Inequality]
If $f : A \to \mathbb{R}$ measurable, non-negative.
For all $\epsilon > 0$,
$$m\left(\{x \in A : f(x) \geq \epsilon\}\right) \leq \frac{1}{\epsilon} \int_A f$$

<u>Proof</u>
Let $\epsilon > 0$ be given and let
$$A_\epsilon = \{x \in A : f(x) \geq \epsilon\}$$

1. $m(A_\epsilon) < \epsilon$.
$$\varphi = \epsilon \chi_{A_\epsilon} \leq f$$
A BF function
$$\epsilon m(A_\epsilon) = \int_A \varphi \leq \int_A f$$

2. $m(A_\epsilon) = \infty$.
For $n \in \mathbb{N}$, $A_{\epsilon,n} := A_\epsilon \cap [-n, n]$.
By the continuity of measurable,
$$\infty = m(A_\epsilon) = \lim_{n \to \infty} m(A_{\epsilon,n}).$$
For $n \in \mathbb{N}$, $\varphi_n := \epsilon \chi_{A_{\epsilon,n}}$ (BF).
We see that $\varphi_n \leq f$.
Therefore,
$$\begin{aligned} \infty &= m(A_\epsilon) \\ &= \lim_{n \to \infty} m(A_{\epsilon,n}) \\ &= \lim_{n \to \infty} \frac{1}{\epsilon} \int_A \varphi_n \\ &\leq \int_A f \end{aligned}$$

<u>Proposition:</u>
$f : A \to \mathbb{R}$ measurable and nonnegative ($f \geq 0$). $\int_A f = 0$ iff $f = 0$ almost everywhere.
<u>Proof:</u>

($\Leftarrow$) suppose $\int_A f = 0$.

$$\begin{aligned} &m(\{x \in A : f(x) \neq 0\}) \\ &\leq \sum m\left(\left\{x \in A : f(x) \geq \frac{1}{n}\right\}\right) \\ &\overset{(CI)}{\leq} \sum n \int_A f = 0 \end{aligned}$$

($\Leftarrow$) Suppose $B = \{x \in A : f(x) \neq 0\}$ has measure 0.

$$\begin{aligned} \int_A f &= \int_B f + \int_{A \setminus B} f \\ &= \int_B f \\ &= 0 \qquad \text{[Piazza]} \end{aligned}$$

6.5 Fatou and MCT

Theorem [Fatou's Lemma]

(f_n) measurable, non-negative, $A \to \mathbb{R}$.
If $f_n \to f$ pointwise, then

$$\int_A f \leq \liminf \int_A f_n$$

Proof
Let $0 \leq h \leq f$ be a BF function. Say $A_0 = \{x \in A : h(x) \neq 0\}$.
It suffices to show

$$\int_A h \leq \liminf \int_A f_n$$

Since h is BF, $m(A_0) < \infty$.
For each $n \in \mathbb{N}$, let

$$h_n = \min\{h, f_n\} \text{ (measurable)}$$

Note:

1. $0 \leq h_n \leq h \leq M$, for some $M > 0$, for all $n \in \mathbb{N}$.
2. For $x \in A_0$ and $n \in \mathbb{N}$,
   (a) $h_n(x) = h(x)$ OR
   (b) $h_n(x) = f_n(x) \leq h(x)$ and

$$\begin{aligned} 0 &\leq h(x) - h_n(x) \\ &= h(x) - f_n(x) \\ &\leq f(x) - f_n(x) \to 0 \end{aligned}$$

Thus, $h_n \to h$ pointwise on A_0.

By the BCT,

$$\begin{aligned} &\lim_{n\to\infty} \int_{A_0} h = \int_{A_0} h \\ \Rightarrow &\lim_{n\to\infty} \int_A h_n = \int_A h \end{aligned}$$

Since $h_n \leq f_n$ on A,

$$\begin{aligned}\int_A h &= \lim_{n\to\infty} \int_A h_n \\ &= \lim_{n\to\infty} \inf \int_A h_n \\ &\leq \lim_{n\to\infty} \inf \int_A f_n\end{aligned}$$

Exercise:

$$A = (0,1]$$

$$f_n = n\chi_{(0,\frac{1}{n})}$$

$f_n \to 0$ pointwise.

$$\int_A 0 = 0$$

$$\int_A f_n = nm(0, \frac{1}{n}) = 1$$

$$\liminf \int_A f_n = 1$$

Theorem [MCT]
(f_n) non-negative, measurable function $A \to \mathbb{R}$.
If (f_n) is increasing and $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Why?

$$\begin{aligned}\int_A f &\overset{\text{FL}}{\leq} \liminf \int_A f_n \\ &\leq \limsup \int_A f_n \\ &\leq \int_A f\end{aligned}$$

Remark:

1. If $\varphi : A \to \mathbb{R}$ is simple and $m(A) < \infty$, then

$$\int_A \varphi < \infty$$

2. If $f : A \to \mathbb{R}$ is bounded, measurable and $m(A) < \infty$, then:

$$\int_A f < \infty$$

Definition:
If $f : A \to \mathbb{R}$ is measurable and $f \geq 0$, then we say f is integrable iff $\int_A f < \infty$.

7 Week 7

7.1 Integration 4

The general integral

Definition:

$f : A \to \mathbb{R}$ measurable,

$$f^+(x) = \max\{f(x), 0\}$$
$$f^-(x) = \max\{-f(x), 0\}$$

Note:

1. $f + f^- = |f|$
2. $f - f^- = f$
3. f^+, f^- measurable.

Proposition

$f : A \to \mathbb{R}$ measurable, then f^+, f^- are integrable iff $|f|$ is integrable.

Why?

($\Rightarrow$)

$|f| = f^+ + f^-$.

$$\int_A |f| = \int_A f^+ + \int_A f^- < \infty$$

($\Leftarrow$)

$$\int_A f^+ \le \int_A |f| < \infty$$
$$\int_A f^- \le \int_A |f| < \infty$$

Definition:

$f : A \to \mathbb{R}$ measurable. We say f is integrable iff $|f|$ is integrable iff f^+, f^- are integrable, and define:

$$\int_A f = \int_A f^+ - \int_A f^-$$
$$(f = f^+ - f^-)$$

Proposition: [Comparison Test]

$f : A \to \mathbb{R}$ measurable, $g : A \to \mathbb{R}$ non-negative integrable.

If $|f| \le g$, then f is integrable and $\left|\int_A f\right| \le \int_A |f|$.

Why?

1. $\int_A |f| \le \int_A f < \infty$.

   f is integrable.

2.

$$\begin{aligned}\left|\int_A f\right| &= \left|\int_A f^+ - \int_A f^-\right| \\ &\le \int_A f^+ + \int_A f^- \\ &= \int_A (f^+ + f^-) = \int_A |f|\end{aligned}$$

Proposition:
$f, g : A \to \mathbb{R}$ integrable.

1. $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable and
$$\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$$
2. If $f \leq g$, then $\int_A f \leq \int_A g$.
3. If $B, C \subseteq A$ are measurable with $B \cap C = \emptyset$, then
$$\int_{B \cup C} f = \int_B f + \int_C f$$

Why?

1. Comparison Test
2. These results hold for f^+, f^-, g^+, g^-.

Theorem [Lebesgue Dominated Convergence Theorem]
(f_n) measurable, $A \to \mathbb{R}$, $f_n \to f$ pointwise.
If there exists an integrable $g : A \to \mathbb{R}$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$, then f is integrable and $\lim_{n\to\infty} \int_A f_n = \int_A f$.
Proof:
Since $|f_n| \leq g \to |f|$ pointwise and so
$$|f| \leq g$$
.
By comparison, f is integrable.
Next, observe that $g - f \geq 0$.
By Fatou's Lemma:

1. $$\begin{aligned}\int_A g - \int_A f &= \int_A g - f \\ &\leq \liminf \int_A g - f_n \\ &= \int_A g - \limsup \int_A f_n \\ &\Rightarrow \limsup \int_A f_n \leq \int_A f\end{aligned}$$
2. $$\begin{aligned}\int_A g + \int_A f &= \int_A g + f \\ &\leq \liminf \int_A g + f_n \\ &= \int_A g + \liminf \int_A f_n \\ &\Rightarrow \int_A f \leq \liminf \int_A f_n\end{aligned}$$

We see that

$$\begin{aligned}\int_A f &= \liminf \int_A f_n \\ &= \limsup \int_A f_n \\ &= \lim_{n\to\infty} \int_A f_n\end{aligned}$$

7.2 Riemann Integration

Definition:
$f : [a, b] \to \mathbb{R}$ bounded.

1. A partition of $[a, b]$ is a finite set:

$$P = \{x_0, x_1, \ldots, x_n\} \subseteq \mathbb{R}$$

   such that,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

2. Relative to P, we define the lower Darboux sum:

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

   where

$$m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$$

3. Similarly, the upper Darboux sum is defined by:

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

   where

$$M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$$

Definition:
$f : [a, b] \to \mathbb{R}$ bounded.

1. Lower Riemann Integral:

$$R\underline{\int_a^b} f = \sup \{L(f, P) : P \text{ partition}\}$$

2. Upper Riemann Integral:

$$R\overline{\int_a^b} f = \inf \{U(f, P) : P \text{ partition}\}$$

3. We say f is Riemann Integrable iff

$$R\underline{\int_a^b} f = R\overline{\int_a^b} f = R\int_a^b f$$

Definition:
Let $I_1, \dots, I_n$ be pairwise disjoint intervals such that:

$$[a,b] = \bigcup_{i=1}^{n} I_i$$

A step function is a function of the form:

$$f = \sum_{i=1}^{n} a_i \chi_{I_i}$$

for some $a_i \in \mathbb{R}$.

Remark:
$f : [a,b] \to \mathbb{R}$ bounded.

$$a = x_0 < x_1 < \cdots < x_n = b$$

$I_i = [x_{i-1}, x_i)$, $i = 1, 2, \dots, n-1$

$$I_n = [x_{n-1}, x_n]$$

Then,

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i \ell(I_i) \\ &= R\int_a^b \varphi \end{aligned}$$

$$\varphi(x) = m_i$$

on I_i, $(\varphi \leq f)$ and

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i \ell(I_i) \\ &= R\int_a^b \psi \end{aligned}$$

$$\psi(x) = M_i$$

on I_i, $(f \leq \psi)$.

Remark:
$f : [a,b] \to \mathbb{R}$ bounded.

$$\begin{aligned} R\underline{\int_a^b} f &= \sup \{L(f,P) : P\} \\ &= \sup \left\{ R\int_a^b \varphi : \varphi \leq f \text{ step} \right\} \end{aligned}$$

$$\begin{aligned} R\overline{\int_a^b} f &= \int \{U(f,P) : P\} \\ &\inf \left\{ R\int_a^b \psi : f \leq \psi \text{ step} \right\} \end{aligned}$$

7.3 Riemann vs Lebesgue

Goal: Compare Lebesgue and Riemann Integration for bounded functions $f : [a,b] \to \mathbb{R}$.

Definition: $f : [a,b] \to \mathbb{R}$ bounded. Let $x \in [a,b]$ and $\delta > 0$.

1. $$m_\delta(x) \\ = \inf \{f(x) : x \in (x-\delta, x+\delta) \cap [a,b]\}$$

2. $$M_\delta(x) \\ = \sup \{f(x) : x \in (x-\delta, x+\delta) \cap [a,b]\}$$

3. Lower boundary of f: $$m(x) = \lim_{\delta \to 0} m_\delta(x)$$

4. Upper boundary of f: $$M(x) = \lim_{\delta \to 0} M_\delta(x)$$

5. Oscillation of f: $$\omega(x) = M(x) - m(x)$$

Remark:
$f : [a,b] \to \mathbb{R}$ bounded
The followings are equivalent:

1. f is continuous at $x \in [a,b]$
2. $M(x) = m(x)$
3. $\omega(x) = 0$.

Lemma:
$f : [a,b] \to \mathbb{R}$ bounded.

1. m is measurable.
2. If $\varphi : [a,b] \to \mathbb{R}$ is a step function with $\varphi \leq f$, then $\varphi(x) \leq m(x)$ for all points of continuity of φ.
3. $$R\underline{\int_a^b} f = \int_{[a,b]} m$$

Proof: Appendix

Lemma: $f : [a,b] \to \mathbb{R}$ bounded.

1. M is measurable.
2. If $\psi : [a,b] \to \mathbb{R}$ is a step function with $f \leq \psi$, then $M(x) \leq \psi(x)$ at all points of continuity of ψ.
3. $$R\overline{\int_a^b} f = \int_{[a,b]} M$$

Theorem [Lebesgue]
Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable iff f is continuous almost everywhere. In that case,

$$R\int_a^b f = \int_{[a,b]} f$$

Why?

$$\begin{aligned} R\underline{\int_a^b} f &= \int_{[a,b]} m \\ &\leq \int_{[a,b]} M \\ &= R\overline{\int_a^b} f \end{aligned}$$

We see that f is Riemann integrable if and only if

$$\int_{[a,b]} m = \int_{[a,b]} M \Leftrightarrow \int_{[a,b]} (M - m) = 0$$

$$\Leftrightarrow M = m \text{ almost everywhere}$$

$$\Leftrightarrow \omega = 0 \text{ almost everywhere}$$

$$\Leftrightarrow f \text{ is continuous almost everywhere}$$

If f is continuous almost everywhere: $\Rightarrow f$ is measurable and

$$\begin{aligned} R\underline{\int_a^b} f &= \int_{[a,b]} m \\ &\leq \int_{[a,b]} f \\ &\leq \int_{[a,b]} M \\ &= R\overline{\int_a^b} f \end{aligned}$$

$$\Rightarrow R\int_a^b f = \int_{[a,b]} f$$

Exercise: $f : [0, 1] \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

f is discontinuous on $[0, 1] \Rightarrow f$ is NOT Riemann integrable.
But $f = 0$ almost everywhere and so

$$\int_{[0,1]} f = \int_{[0,1]} 0 = 0$$

Exercise:

$$\mathbb{Q} \cap [0, 1] = \{q_1, q_2, \dots\}$$

$$f_n = \chi_{\{q_1, q_2, \dots, q_n\}}$$

$f_n \to f$ pointwise.

(f_n) increasing. $f_n \le 1$ is Riemann integrable.

$$R \int_{[0,1]} f_n \not\to R \int_{[0,1]} f$$

We do not have MCT, RDCT.

8 Week 8

8.1 L^p Spaces

Goal:

Create Banach Spaces whose norm is given by Lebesgue Integration.

Recall:

1. For $1 \le p < \infty$
$$\left(C\left([a,b]\right), \|\cdot\|_p\right)$$
is a normed vector space, where
$$\|f\|_p^p = \int_a^b |f|^p$$

2. For $p = \infty$,
$$(C([a,b]), \|\cdot\|_\infty)$$
$$\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$$
is a Banach space. (A complete normed vector space)

Problem: $A \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$.

$$\|f\|_p = \left(\int_A |f|^p\right)^{1/p}$$

is not a norm on the vector space of integrable functions $f : A \to \mathbb{R}$

Why?

$$\int_A |f|^p = 0 \Leftrightarrow f = 0 \text{ almost everywhere}$$

Definition / Notation

$A \subseteq \mathbb{R}$ measurable.

1. $\mathcal{M}(A) = \{f : A \to \mathbb{R} \text{ measurable}\}$ is a vector space.

$$f \sim g \text{ iff } f = g \text{ almost everywhere}$$
$$[f] \text{ equivalence class}$$

2. $\mathcal{M}(A)/\sim = \{[f] : f \in \mathcal{M}(A)\}$
$$\alpha[f] + \beta[g] = [\alpha f + \beta g]$$
is a vector space.

Remark [Piazza]:
If $f \sim g$ and f is integrable, then g is integrable and $\int_A f = \int_A g$
Definition:
$A \subseteq \mathbb{R}$ measurable, $1 \leq p < \infty$.

$$L^p(A) = \left\{ [f] \in \mathcal{M}(A)/\sim : \int_A |f|^p < \infty \right\}$$

Remark
Suppose $[f], [g] \in L^p(A)$. Then,

$$\int_A |f|^p, \int_A |g|^p < \infty$$

1.

$$\begin{aligned} |f+g|^p &\leq (|f| + |g|)^p \\ &\leq (2 \max \{|f|, |g|\})^p \\ &\leq 2^p \left(|f|^p + |g|^p\right) \end{aligned}$$

$\Rightarrow |f+g|^p$ integrable by comparison.

2. $L^p(A)$ is a subspace of $\mathcal{M}(A)/\sim$.

Definition:
$A \subseteq \mathbb{R}$ measurable.

$$L^\infty(A) = \{[f] \in \mathcal{M}(A)/\sim : f \text{ bounded almost everywhere}\}$$

Remark:

1. $[f], [g] \in L^\infty(A)$
   $|f| \leq M$ off $B \subseteq A, m(B) = 0$
   $|g| \leq N$ off $C \subseteq A, m(C) = 0$
   For $x \notin B \cup C$, ($B \cup C$ has measure 0),

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N$$

2. $L^\infty(A)$ is a subspace of $\mathcal{M}(A)/\sim$.

Proposition: $A \subseteq \mathbb{R}$ be measurable.
Then,

$$\|[f]\|_\infty = \inf \{M \geq 0 : |f| \leq M \text{ almost everywhere}\}$$

is a norm on $L^\infty(A)$.
Remark:
For all $n \in \mathbb{N}$,

$$|f| \leq \|[f]\|_\infty + \frac{1}{n}$$

off $m(A_n) = 0$.

$$B = \bigcup_{n=1}^{\infty} A_n \rightarrow \text{ measure } 0$$

$$|f| \leq \|[f]\|_\infty \text{ off } B$$

Why?

1. $\|[f]\|_\infty = 0 \Rightarrow |f| \leq \|[f]\|_\infty$ almost everywhere.
   $\Rightarrow |f| = 0$ almost everywhere.
   $\Rightarrow f = 0$ almost everywhere.
   $[f] = [0]$ in $L^\infty(A)$.

2. $|f| \leq \|[f]\|_\infty$ off B.
   $|g| \leq \|[g]\|_\infty$ off C.
   Both B and C have measure 0.
   Off $B \cup C \to$ measure 0:

$$\begin{aligned} |f+g| &\leq |f| + |g| \\ &\leq \|[f]\|_\infty + \|[g]\|_\infty \end{aligned}$$

   By the definition of inf,

$$\begin{aligned} \|[f+g]\|_\infty &= \|[f]+[g]\|_\infty \\ &\leq \|[f]\|_\infty + \|[g]\|_\infty \end{aligned}$$

8.2 L^p Norm

Goal

Show that

$$\|[f]\|_p = \left(\int_A |f|^p\right)^{1/p}$$

is a norm on $L^p(A)$, for $1 \leq p < \infty$.

Example: $p = 1$:

$A \subseteq \mathbb{R}$ measurable, $[f], [g] \in L^1(A)$

$$\begin{aligned} |f+g| &\leq |f| + |g| \\ \Rightarrow \int_A |f+g| &\leq \int_A |f| + \int_A |g| \\ \Rightarrow \|[f+g]\|_1 &\leq \|[f]\|_1 + \|[g]\|_1 \end{aligned}$$

Remember:

$f = g$ in $L^p(A)$ means $f = g$ almost everywhere.

Definition:

For $p \in (1, \infty)$, we define $q = \frac{p}{p-1}$ to be the Holder conjugate of p.

Note:

1. $q = \frac{p}{p-1} \Leftrightarrow p = \frac{q}{q-1}$

2. $\frac{1}{p} + \frac{1}{q} = 1$

Definition:

We define 1 and ∞ to be Holder conjugates.

Proposition: [Young's Inequality]

$p, q \in (1, \infty)$ be Holder conjugate:

For all $a, b > 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^p}{p}$$

Why?

$$f(x) = \frac{1}{p}x^p + \frac{1}{q} - x \text{ on } (0, \infty)$$

$$f'(x) = x^{p-1} - 1$$

$$f(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$$

$$\Rightarrow f \geq 0 \text{ on } (0, \infty)$$

$$\Rightarrow x \leq \frac{1}{p}x^p + \frac{1}{q} \forall x > 0$$

Taking:

$$x = \frac{a}{b^{q-1}}$$

$$\Rightarrow \frac{a}{b^{q-1}} \leq \frac{1}{p} \cdot \frac{a^p}{b^{(q-1)p}} + \frac{1}{q}$$

$$\Rightarrow \frac{a}{b^{q-1}} \leq \frac{1}{p} \cdot \frac{a^p}{b^q}$$

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

<u>Proposition:</u> [Holder's Inequality]
$A \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q is the Holder Conjugate.
If $f \in L^p(A)$ and $g \in L^q(A)$, then $fg \in L^1(A)$ and

$$\int_A |fg| \leq \|f\|_p \|g\|_q$$

<u>Why?</u>

1. $p = 1, q = \infty$:

$$\begin{aligned} |fg| &= |f| \cdot |g| \\ &\leq |f| \cdot \|g\|_\infty \text{ almost everywhere} \end{aligned}$$

   Integrable by Comparison.

$$\Rightarrow fg \in L^1(A)$$

$$\int_A |fg| \leq \int_A |f| \cdot \|g\|_\infty = \|g\|_\infty \|f\|_1$$

2. $1 < p < \infty$, q is the Holder Conjugate.

$$|fg| = |f| \cdot |g| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q}$$

   is integrable by comparison.

$$fg \in L^1(A)$$

Also,

$$\int_A |fg| \le \frac{1}{p}\int_A |f|^p + \frac{1}{q}\int_A |g|^q$$
$$= \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q$$

(a) $\|f\|_p = \|g\|_q = 1$

$$\int_A |fg| \le \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \cdot \|g\|_q$$

(b) $\frac{f}{\|f\|_p}, \frac{g}{\|g\|_q}$
By (a),

$$\frac{1}{\|f\|_p\|g\|_q}\int_A |fg| \le 1$$
$$\Rightarrow \int_A |fg| \le \|f\|_p \cdot \|g\|_q$$

Lemma
p, q are Holder Conjugate, $f \in L^p(A)$.
If $f \neq 0$,

$$f^* = \|f\|_p^{1-p}\,\mathrm{sgn}(f)|f|^{p-1}$$

is in $L^q(A)$ and

$$\int_A ff^* = \|f\|_p$$
$$\|f^*\|_q = 1$$

Why?

1. $p = 1$, $q = \infty$.

$$f^* = \mathrm{sgn}(f) \in L^\infty(A)$$
$$\int_A ff^* = \int_A |f| = \|f\|_1$$

$$\|f^*\|_\infty = 1$$

2. $1 < p < \infty$, q is the Holder Conjugate.

$$\int_A ff^* = \|f\|_p^{1-p}\int_A |f|^p = \|f\|_p^{1-p}\|f\|_p^p$$
$$= \|f\|_p$$

$$\|f^*\|_q^q = \|f\|_p^{(1-p)q}\int_A |f|^{(p-1)q}$$
$$= \|f\|_p^{-p}\int_A |f|^p$$
$$= \|f\|_p^{-p}\|f\|_p^p = 1$$

Theorem [Minkowski's Inequality]
$A \subseteq \mathbb{R}$ measurable, $1 \le p < \infty$.
If $f, g \in L^p(A)$, then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

Proof:

1. $p = 1$ Done.

2. $1 < p < \infty$.

$$\begin{aligned}
\|f+g\|_p &= \int_A (f+g)(f+g)^* \\
&= \int_A f(f+g)^* = \int_A g(f+g)^* \\
&\overset{H}{\le} \|f\|_p \cdot \|(f+g)^*\|_q + \|g\|_p \cdot \|(f+g)^*\|_q \\
&= \|f\|_p + \|g\|_p
\end{aligned}$$

8.3 Completeness

Goal:
Prove that $L^p(A)$ is a Banach space for all $1 \le p \le \infty$.
Theorem [Riesz-Fisher]
For every measurable $A \subseteq \mathbb{R}$ and $1 \le p \le \infty$, $L^p(A)$ is a Banach space.
Proof:

1. $p = \infty$, Piazza.

2. $1 \le p < \infty$
   Let $(f_n) \subseteq L^p(A)$ be strongly-Cauchy. Therefore, there exists $(\epsilon_n) \subseteq \mathbb{R}$ such that:

   (a) $\|f_{n+1} - f_n\|_p \le \epsilon_n^2$
   (b) $\sum \epsilon_n < \infty$

   Idea: Since $\mathbb{R}$ is complete, if $(f_n(x))$ is strongly-Cauchy, then it converges.

   For each $n \in \mathbb{N}$,

$$\begin{aligned}
A_n &:= \{x \in A : |f_{n+1}(x) - f_n(x)| \ge \epsilon\} \\
&= \{x \in A : |f_{n+1}(x) - f_n(x)|^p \ge \epsilon_n^p\}
\end{aligned}$$

   By Chebychev:

$$m(A_n) \le \frac{1}{\epsilon_n^p} \int_A |f_{n+1} - f_n|^p \le \frac{1}{\epsilon_n^p} \epsilon_p^{2p} = \epsilon_n^p$$

$$\Rightarrow \sum m(A_n) \le \sum \epsilon_n^p \le \left(\sum \epsilon_n\right)^p < \infty$$

$$m(\limsup A_n) = 0$$

Fix $x \notin \limsup A_n$.

Let $N = \max\{n : x \in A_n\}$

For $n > N$,

$$|f_{n+1}(x) - f_n(x)| < \epsilon_n^2, \sum \epsilon_n < \infty$$

$$\Rightarrow (f_n(x)) \text{ Cauchy}$$
$$f_n(x) \to f(x) \in \mathbb{R}$$

$f_n \to f$ pointwise almost everywhere.

For $k \in \mathbb{N}$,

$$\begin{aligned}\|f_{n+k} - f_n\|_p &\le \|f_{n+k} - f_{n+k-1}\|_p + \cdots + \|f_{n+1} - f_n\|_p \\ &\le \epsilon_{n+k-1}^2 + \cdots + \epsilon_n^2 \\ &\le \sum_{i=1}^{\infty} \epsilon_i^2\end{aligned}$$

$|f_{n+k} - f_n|^p \to |f_n - f|^p$ pointwise almost everywhere as $k \to \infty$.

By Fatou,

$$\begin{aligned}&\int_A |f_n - f|^p \\ \le &\liminf_{k\to\infty} \int_A |f_{n+k} - f_n|^p \\ = &\liminf_{k\to\infty} \|f_{n+k} - f_n\|_p^p \\ \le &\left[\sum_{i=n}^{\infty} \epsilon_i^2\right]^p \to 0\end{aligned}$$

8.4 Separability

Goal: Prove that $L^p(A)$ is separable for all $1 \le p < \infty$.

Recall:

A metric space X is separable if it has a countable, dense subset.

Exercise:

$p = \infty$?

Suppose $\{f_n : n \in \mathbb{N}\}$ is dense in $L^\infty[0,1]$.

For every $x \in [0,1]$, we may find

$$\|\chi_{[0,x]} - f_{\theta(x)}\|_\infty < \frac{1}{2}$$

For $x \neq y$ in $[0,1]$

$$\|\chi_{[0,x]} - \chi_{[0,y]}\|_\infty = 1$$

$\theta : [0,1] \to \mathbb{N}$ is injective.

Contradiction.

Notation:

$\text{Simp}(A)$ = simple functions on measurable set A.

$\text{Step}[a,b]$ = step functions on $[a,b]$.

$\text{Step}_{\mathbb{Q}}[a,b]$ = step functions on $[a,b]$, with rational partition and function values.

$\text{Step}_{\mathbb{Q}}[a,b]$ countable.

Proposition:
$A \subseteq \mathbb{R}$ measurable, $1 \leq p < \infty$.
Simp(A) is dense in $L^p(A)$.
Why?
$f \in L^p(A) \to f$ measurable
There exists (ψ_n) simple functions:

1. $\varphi_n \to f$ pointwise.
2. $|\varphi_n| \leq |f| \to |\varphi_n|^p \leq |f|^p$

By Comparison, $(\varphi_n) \subseteq L^p(A)$
Note:

$$\|\varphi_n - f\|_p^p = \int_A |\varphi_n - f|^p$$

$$\begin{aligned} |\varphi_n - f|^p &\leq 2^p \left(|\varphi_n|^p + |f|^p\right) \\ &\leq 2^{p+1}|f|^p \end{aligned}$$

By the Lebesgue Dominated Convergence Theorem:

$$\lim_{n\to\infty} \int_A |\varphi_n - f|^p = \int_A 0 = 0$$

Fact: This is also true for $p = \infty$.
Proposition:
$1 \leq p < \infty$
Step$[a, b]$ is dense in $L^p[a, b]$
Why?
$A \subseteq [a, b]$ measurable, $\chi_A : [a, b] \to \mathbb{R}$.
Littlewood 1:

$$\exists \bigsqcup_{i=1}^{n} I_i = U (I_i \text{ being bounded, open interval})$$

$$m\left(U \triangle A\right) < \boxed{\text{Stuff}}$$

$$\chi_U : [a, b] \to \mathbb{R} \text{ (Step functions)}$$

$$\begin{aligned} &\|\chi_U - \chi - A\|_p^p \\ =&\int_A |\chi_U - \chi_A|^p \\ =&m\left(A \triangle U\right) \end{aligned}$$

$\Rightarrow \|\chi_U - \chi_A\|_p < \epsilon$.
Corollary:
$1 \leq p < \infty$.
Step$_{\mathbb{Q}}[a, b]$ is dense in $L^p[a, b]$.
Therefore, $L^p[a, b]$ is separable.

Proposition:
$1 \leq p < \infty$.
$L^p(\mathbb{R})$ is separable.
Why?

$$F_n = \left\{ f \in L^p(\mathbb{R}) : f|_{[-n,n]} \in \text{Step}_{\mathbb{Q}}[-n,n], f|_{\mathbb{R}\setminus[-n,n]} = 0 \right\}$$

$F = \bigcup_{n=1}^{\infty} F_n$ countable.
Take $f \in L^p(\mathbb{R})$. Fix $n \in \mathbb{N}$.

$$\Rightarrow f|_{[-n,n]} \in L^p\left([-n,n]\right)$$

We show

$$f\chi_{[-n,n]} \to f$$

in $L^p(\mathbb{R})$
Note:

1. $$\begin{aligned} &\|f\chi_{[-n,n]} - f\|_p^p \\ &= \int_R |f\chi_{[-n,n]} - f|^p \\ &= \int_{\mathbb{R}\setminus[-n,n]} |f|^p \\ &= \int_{\mathbb{R}} |f|^p \chi_{\mathbb{R}\setminus[-n,n]} \end{aligned}$$

2. $$\left| |f|^p \chi_{\mathbb{R}\setminus[-n,n]} \right| \leq |f|^p$$

   integrable

3. By the LDCT,

   $$\begin{aligned} &\lim_{n\to\infty} \|f\chi_{[-n,n]} - f\|_p^p \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} |f\chi_{[-n,n]} - f|^p \\ &= \int_{\mathbb{R}} 0 = 0 \end{aligned}$$

   Therefore, $\|f\chi_{[-n,n]} - f\|_p \to 0$.

   For each $n \in \mathbb{N}$, there exists $\varphi_n \in F$ such that

   $$\|f\chi_{[-n,n]} - \varphi_n\|_p < \frac{1}{n}$$

   Therefore, $\|\varphi_n - f\|_p \to 0$

Theorem
$1 \leq p < \infty$, $A \subseteq \mathbb{R}$ measurable.
$L^p(A)$ is measurable.
Why?
F as before:
$\{f|_A : f \in F\}$ is a countable dense subset of $L^p(A)$.

9 Week 9

9.1 Hilbert Spaces

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Definition:

V is a vector space over $\mathbb{F}$. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that

1. For all $v \in V$, $\langle v, v \rangle \in \mathbb{R}$, $\langle v, v \rangle \geq 0$ with $\langle v, v \rangle = 0$ iff $v = 0$.
2. For all $v, w \in V$, $\langle v, w \rangle = \overline{\langle w, v \rangle}$
3. For all $\alpha \in \mathbb{F}$, $u, v, w \in V$:
$$\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$$

We call $(V, \langle \cdot, \cdot \rangle)$ an inner product space.

Proposition:

Let V be an inner product space.

$$\|u\| = \sqrt{\langle v, v \rangle}$$

is a norm on V. We call $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$

Example:

$A \subseteq \mathbb{R}$ measurable. $V = L^2(A)$.

$$\langle f, g \rangle = \int_A fg$$

is an inner product space.

Note:

$$\sqrt{\langle f, f \rangle} = \left(\int_A |f|^2 \right)^{1/2} = \|f\|_2$$

Exercise:

$A \subseteq \mathbb{R}$ measurable.

$$V = L^2 (A, \mathbb{C})$$

[See A3]

$$\langle f, g \rangle = \int_A f\overline{g}$$

$$\sqrt{\langle f, f \rangle} = \|f\|_2$$

Proposition: [Parallelogram Law]

Let V be an inner product space. For all $u, v \in V$,

$$\|u + v\|^2 + \|u - v\|^2 = 2 \left(\|u\|^2 + \|v\|^2 \right)$$

Why?

$$\begin{aligned}
&\|u + v\|^2 + \|u - v\|^2 \\
=&\langle u + v, u + v \rangle + \langle u - v, u - v \rangle \\
=&\langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle + \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\
=&2 \left(\langle u, u \rangle + \langle v, v \rangle \right) \\
=&2 \left(\|u\|^2 + \|v\|^2 \right)
\end{aligned}$$

Example:

Let $1 \le p < \infty$, $V = L^p[0,2]$. $f = \chi_{[0,1]}, g = \chi_{[1,2]}$.

$$\begin{aligned} \|f\|_p^2 &= \left(\int_{[0,2]} |f|^p\right)^{2/p} \\ &= 1^{2/p} = 1 \end{aligned}$$

$$\begin{aligned} \|g\|_p^2 &= 1^{2/p} = 1 \\ \|f+g\|_p^2 &= 2^{2/p} \\ \|f-g\|_p^2 &= 2^{2/p} \end{aligned}$$

We get the parallelogram law:

$$\begin{aligned} &\Leftrightarrow 2^{2/p} + 2^{2/p} + 2(1+1) \\ &\Leftrightarrow 2^{2/p} = 2 \Leftrightarrow p = 2 \end{aligned}$$

Therefore, $\|\cdot\|_p$ is induced by an inner product iff $p = 2$.
[Piazza] $\|\cdot\|_\infty$ is NOT induced by an inner product.
Definition:
A Hilbert Space is a complete inner product space. (i.e., a Banach space whose norm is induced by an inner product.)
Examples:
$L^2(A), L^2(A, \mathbb{C})$ are Hilbert spaces.

9.2 Orthogonality

Definition:
Let V be an inner product space. We say $v, w \in V$ are orthogonal if $\langle v, w \rangle = 0$.
Example
$f, g \in L^2([-\pi,\pi], \mathbb{C})$, $m \ne n, m, n \in \mathbb{Z}$. $f(x) = e^{inx}, g(x) = e^{imx}$

$$\begin{aligned} \langle f, g \rangle &= \int_{[-\pi,\pi]} f\overline{g} \\ &= \int_{[-\pi,\pi]} e^{inx}e^{-imx}dx \\ &= \int_{[-\pi,\pi]} e^{ix(n-m)}dx \\ &= \int_{[-\pi,\pi]} \cos((n-m)x) + i\int_{[-\pi,\pi]} \sin((n-m))x \\ &= R\int_{-\pi}^{\pi} \cos((n-m)x) + R\int_{-\pi}^{\pi} \sin((n-m)x)\,dx \\ &= \left[\frac{1}{n-m}\sin((n-m)x)\right]_{-\pi}^{\pi} + \left[\frac{-1}{n-m}\cos((n-m)x)\right]_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

Theorem [Pythagorean Theroem]
Let V be an inner product space.
If $v_1, \ldots, v_n \in V$ are pairwise orthogonal, then

$$\left\|\sum v_i\right\|^2 = \sum \|v_i\|^2$$

Definition:

Let V be an inner product space. We say $A \subseteq V$ is orthonormal if the elements of A are pairwise orthogonal and $\|v\| = 1$ for all $v \in A$.

Corollary:

Let V be an inner product space, $\{v_1, \ldots, v_n\}$ orthonormal.

$$\left\|\sum \alpha_i v_i\right\|^2 = \sum |\alpha_i|^2$$

where $\alpha_i \in \mathbb{R}$.

Exercise:

$L^2([-\pi, \pi], \mathbb{C})$

$A = \left\{\frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z}\right\}$ is pairwise orthogonal.

$$\begin{aligned}\frac{1}{2\pi} \left\|e^{inx}\right\|_2^2 & \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{inx} e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi]} 1 \\ &= 1 \Rightarrow A \text{ is orthonormal}\end{aligned}$$

Definition:

Let V be an inner product space.

An orthonormal basis is a maximal (with respect to $\subseteq$) orthonormal subset of V.

Fact: An inner product space always has an orthonormal basis.

Fact: Let H be a Hilbert Space. If $W \subseteq H$ is closed subspace, then there exists a subspace $W^\perp \subseteq H$ such that

$$H = W \oplus W^\perp$$

and $\langle w, z \rangle = 0$ for all $w \in W$ and $z \in W^\perp$.

Theorem

Let H be a Hilbert Space, then H has a countable orthonormal basis iff H is separable.

Proof:

- Forward Direction:

  Let B be a countable orthonormal basis for H.

  Claim:

  $W = \text{Span}(B)$, $\overline{W} = H$.

  Suppose $\overline{W} \neq H$. Since $H = \overline{W} \oplus \overline{W}^\perp$, we may find $0 \neq x \in \overline{W}^\perp$. We may assume $\|x\| = 1$.

  Therefore, $B \cup \{x\}$ is orthonormal.

  Contradiction.

  Therefore, $\overline{W} = H$.

  $\Rightarrow \overline{\text{Span}_{\mathbb{Q}}(B)} = H$ is a countable set.

  Therefore, H is separable.

- Backwards Direction:

  Suppose H does not have an orthonormal basis, which is countable.

  Let B be an orthonormal basis for H.

Therefore, B is uncountable.

For $u \neq v$ in B,

$$\begin{aligned}\|u-v\|^2 = \|u\|^2 + \|v\|^2 = 2 \\ \Rightarrow \|u-v\| = \sqrt{2}\end{aligned}$$

Suppose $X \subseteq H$ such that $\overline{X} = H$.

For every $u \in B$, there exists $x_u \in X$ such that

$$\|u - x_u\| < \frac{\sqrt{2}}{2}$$

For $u \neq v$ in B, we have that $x_u \neq x_v$.

Therefore, $\varphi : B \to X$, $\varphi(u) = x_u$ is an injection.

Exercise:

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z} \right\}$$

is a countable orthonormal set in $L^2([-\pi, \pi], \mathbb{C})$.

Countable, Orthonormal, Maximal ???

9.3 Big Theorems

Remark

Let H be an inner product space. Let $\{v_1, v_2, \ldots, v_n\}$ be orthonormal.
If $v = \sum \lambda_i v_i$, then

$$\lambda_i = \langle v, v_i \rangle$$

We call $\langle v, v_i \rangle$ the Fourier coefficient of v with respect to $\{v_1, v_2, \ldots, v_n\}$

Definition:

Let H be Hilbert Spaces, $\{v_1, v_2, \ldots\}$ be an orthonormal set. For $v \in H$, we call:

$$\sum_{i=1}^{\infty} \langle v, v_i \rangle v_i$$

the Fourier series of v relative to $\{v_1, v_2, \ldots\}$ and write:

$$v \sim \sum_{i=1}^{\infty} \langle v, v_i \rangle v_i$$

- Converges?
- Converges to v?

Theorem [Best Approximation]

Let H be Hilbert Space, $\{v_1, \ldots, v_n\}$ be a finite orthonormal set in H.
For $v \in H$, $\|v - \sum \lambda_i v_i\|$ is minimized when

$$\lambda_i = \langle v, v_i \rangle$$

Moreover,

$$\begin{aligned}&\|v - \sum \langle v, v_i \rangle v_i\|^2 \\ =&\|v\|^2 - \sum |\langle v, v_i \rangle|^2\end{aligned}$$

Why?

1. $W = \text{Span}\{v_1, \dots, v_n\}$ closed.
   $V = W \oplus W^{\perp}$.

2. $x \in W$. $v = w + z, w \in W, z \in W^{\perp}$.

$$\begin{aligned}\|v - x\|^2 &= \|w + z - x\|^2 \\ &= \|w - x + z\|^2 \\ &= \|w - x\|^2 + \|z\|^2 \\ &\geq \|z\|^2 = \|v - w\|^2\end{aligned}$$

   $\Rightarrow \|v - x\| \geq \|v - w\|$

3. $v = \sum \lambda_i v_i + z, z \in W^{\perp}$.

$$\begin{aligned}\langle v, v_i \rangle &= \lambda_i + \langle z, v_i \rangle \\ &= \lambda_i\end{aligned}$$

4. $v = \sum \langle v, v_i \rangle v_i + z, z \in W^{\perp}$

$$\begin{aligned}\Rightarrow \|v\|^2 &= \left\| \sum \langle v, v_i \rangle v_i \right\|^2 + \|z\|^2 \\ &= \sum |\langle v, v_i \rangle|^2 + \|z\|^2\end{aligned}$$

   Therefore,

$$\begin{aligned}&\left\| v - \sum \langle v, v_i \rangle \right\|^2 \\ =&\|z\|^2 \\ =&\|v\|^2 - \sum |\langle v, v_i \rangle|^2\end{aligned}$$

<u>Theroem</u> [Bessel's Inequality]
Let H be Hilbert Space, $\{v_1, v_2, \dots, v_n\}$ is orthonormal.
If $v \in H$,

$$\sum_{i=1}^{n} |\langle v, v_i \rangle|^2 \leq \|v\|^2$$

<u>Why?</u>

$$\|v\|^2 - \sum |\langle v, v_i \rangle|^2 = \|?\|^2 \geq 0$$

<u>Theorem</u> [Parseval's Identity]
Let H be a Hilbert Space, $\{v_1, v_2, v_3, \dots\}$ be a countable orthonormal set.
For $v \in H$,

$$\sum_{i=1}^{\infty} |\langle v, v_i \rangle|^2 = \|v\|^2$$

iff

$$\lim_{n\to\infty}\left\|v-\sum_{i=1}^{n}\langle v,v_i\rangle v_i\right\|=0$$

Theorem [Orthonormal Basis Test]
Let H be a separable Hilbert Space, $\{v_1, v_2, \dots\}$ be orthonormal.
The followings are equivalent:

1. $\{v_1, v_2, \dots\}$ is basis.
2. $\overline{\text{Span}\{v_1, v_2, \dots\}} = H$
3. $\lim_{n\to\infty}\|v-\sum_{i=1}^{n}\langle v,v_i\rangle v_i\|=0$ for every $v \in H$.

Why?
(1) $\Rightarrow$ (2): Done.
(2) $\Rightarrow$ (1):
If $\{v_1, v_2, \dots\}$ is not maximal, then we may find $u \in H$, $\|u\| = 1$ such that $\langle u, v_i\rangle = 0$, $\forall i \in \mathbb{N}$.
Since $C = \{x \in H : \langle x, u\rangle = 0\}$ is closed, $u \notin \overline{\text{Span}\{v_1, v_2, \dots\}}$.
(2) $\Rightarrow$ (3):
Let $v \in H$ and let $\epsilon > 0$ be given:
Let $\sum_{i=1}^{N}\alpha_i v_i \in \text{Span}\{v_1, \dots\}$ such that:

$$\left\|v-\sum_{i=1}^{N}\alpha_i v_i\right\|<\epsilon$$

Therefore, $\|v-\sum_{i=1}^{N}\langle v,v_i\rangle v_i\|<\epsilon$.
For $n \geq N$,

$$\begin{aligned}&\left\|v-\sum_{1}^{n}\langle v,v_i\rangle v_i\right\|\\ \leq&\left\|v-\sum_{1}^{N}\langle v,v_i\rangle v_i\right\|+\left\|\sum_{N+1}^{n}\langle v,v_i\rangle v_i\right\|\\ <&\epsilon+\sqrt{\sum_{N+1}^{\infty}|\langle v,v_i\rangle|^2}\\ \to&0 \text{ as } N\to\infty\end{aligned}$$

(3) $\Rightarrow$ (2):
Similar.

10 Week 10

10.1 Fourier Series

Motivating Questions:

1. Is $\left\{\frac{1}{\sqrt{2\pi}}e^{inx} : n \in \mathbb{Z}\right\}$ an orthonormal basis for $L^2([-\pi,\pi],\mathbb{C})$?
2. Is $\text{Span}\{e^{inx} : n \in \mathbb{Z}\}$ dense in $L^2([-\pi,\pi],\mathbb{C})$?
3. Is $\text{Span}\{e^{inx} : n \in \mathbb{Z}\}$ dense in $L^1([-\pi,\pi],\mathbb{C})$?

Pictorially:
Given $f \in L^1([-\pi, \pi])$
Can we approximate f using sinusoidal functions?
Definition:
Let $T = [-\pi, \pi)$. We call T the torus or the circle.
We define:

$$L^p(T) := L^p([-\pi, \pi), \mathbb{C})$$

for $1 \leq p < \infty$.
Using the norm,

$$\|f\|_p = \left(\frac{1}{2\pi}\int_T |f|^p\right)^{1/p}$$

$L^p(T)$ is a separable Banach space.
Remark

1. As a group under addition modulo 2π,

$$T \cong \mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C} : |z| = 1\}$$

2. In this way, T is a locally compact abelian group.
3. There is a one-to-one correspondence between $f : T \to \mathbb{C}$ and 2π-periodic functions $f : \mathbb{R} \to \mathbb{C}$.

Definition:
Let $f \in L^1(T)$.

1. We define the nth ($n \in \mathbb{Z}$) Fourier coefficient of f by:

$$\langle f, e^{inx}\rangle := \frac{1}{2\pi}\int_T f(x)e^{-inx}\,dx$$

2. We define the Fourier series of f by:

$$f \sim \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

   where $a_n = \langle f, e^{inx}\rangle$.

3. We let

$$S_N(f, x) = \sum_{-N}^{N} a_n e^{inx}$$

   denote the Nth partial sum of the above Fourier series.

Proposition
Consider the trigonometric polynomial $f \in L^1(T)$ given by:

$$f(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

for some $a_i \in \mathbb{C}$.
For each $-N \leq n \leq N$,

$$\langle f, e^{inx}\rangle = a_n$$

Why?

$$\frac{1}{2\pi}\int_T e^{imx}e^{-inx}\,dx = \delta_{m,n}$$

Remark
Suppose $f \in L^1(T)$ is real-valued.

$$f \sim \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

For $N \in \mathbb{N}$,

$$\begin{aligned}
S_N(f,x) &= \sum_{n=-N}^{N} a_n e^{inx} \\
&= a_0 + \sum_{n=1}^{N}\left(a_n e^{inx} + a_{-n}e^{-inx}\right) \\
&= a_0 + \sum_{n=1}^{N}\left((a_n + a_{-n})\cos(nx) + i\,(a_n - a_{-n})\sin(nx)\right) \\
&= a_0 + \sum_{n=1}^{N} b_n\cos(nx) + c_n\sin(nx)
\end{aligned}$$

Now,

$$a_0 = \frac{1}{2\pi}\int_T f(x)e^{-i0x}\,dx = \frac{1}{2\pi}\int_T f(x)\,dx$$

$$\begin{aligned}
b_n &= a_n + a_{-n} \\
&= \frac{1}{2\pi}\int_T f(x)\left(e^{-inx} + e^{inx}\right)\,dx \\
&= \frac{1}{\pi}\int_T f(x)\cos(nx)\,dx
\end{aligned}$$

$$\begin{aligned}
c_n &= i\,(a_n - a_{-n}) \\
&= \frac{i}{2\pi}\int_T f(x)\left(e^{-inx} - e^{inx}\right)\,dx \\
&= \frac{1}{\pi}\int_T f(x)\sin(nx)\,dx
\end{aligned}$$

are all real-valued.

10.2 Fourier Coefficients

Proposition
$f, g \in L^1(T)$.

1. $\langle f+g, e^{inx}\rangle = \langle f, e^{inx}\rangle + \langle g, e^{inx}\rangle$
2. For $\alpha \in \mathbb{C}$, $\langle \alpha f, e^{inx}\rangle = \alpha\langle f, e^{inx}\rangle$
3. If $\overline{f} : T \to \mathbb{C}$ is defined by $\overline{f}(x) = \overline{f(x)}$, then $\overline{f} \in L^1(T)$ and $\langle \overline{f}, e^{inx}\rangle = \overline{\langle f, e^{-inx}\rangle}$

Why?
(3): $\|f\| = \|\overline{f}\| \Rightarrow \overline{f} \in L^1(T)$

$$
\begin{aligned}
&\langle \overline{f}, e^{inx} \rangle \\
=&\frac{1}{2\pi}\int_T \overline{f}(x)e^{-inx}\,dx \\
=&\frac{1}{2\pi}\int_T \overline{f(x)e^{inx}}\,dx \\
=&\frac{1}{2\pi}\int_T \mathrm{Re}\left(\overline{f(x)e^{inx}}\right)\,dx + \frac{i}{2\pi}\int_T \mathrm{Im}\left(\overline{f(x)e^{inx}}\right)\,dx \\
=&\frac{1}{2\pi}\int_T \mathrm{Re}\left(f(x)e^{inx}\right)\,dx - \frac{i}{2\pi}\int_T \mathrm{Im}(f(x)e^{inx})\,dx \\
=&\overline{\frac{1}{2\pi}\int_T f(x)e^{inx}\,dx} \\
=&\overline{\langle f, e^{-inx} \rangle}
\end{aligned}
$$

Proposition
Let $f \in L^1(T)$, $\alpha \in \mathbb{R}$.
By a previous remark, we may view $f : \mathbb{R} \to \mathbb{C}$ as a 2π-periodic function which is integrable over T.
For $\alpha \in \mathbb{R}$, $f_\alpha : \mathbb{R} \to \mathbb{C}$ given by $f_\alpha(x) = f(x - \alpha)$ is integrable over T and $\langle f_\alpha, e^{inx} \rangle = \langle f, e^{inx} \rangle e^{-in\alpha}$
Proof:
Assignment.
Proposition
$f \in L^1(T)$, for all $n \in \mathbb{Z}$, $|\langle f, e^{inx} \rangle| \le \|f\|_1$.
Proof:

$$
\begin{aligned}
|\langle f, e^{inx} \rangle| &= \left|\frac{1}{2\pi}\int_T f(x)e^{-inx}\,dx\right| \\
&\le \frac{1}{2\pi}\int_T \left|f(x)e^{-inx}\right|\,dx \\
&= \frac{1}{2\pi}\int_T |f(x)|\,dx \\
&= \|f\|_1
\end{aligned}
$$

Corollary $f_k \to f$ in $L^1(T)$.
For all $n \in \mathbb{Z}$,

$$\langle f_k, e^{inx} \rangle \to \langle f, e^{inx} \rangle$$

Proof:

$$
\begin{aligned}
&\left|\langle f_k, e^{inx} \rangle - \langle f, e^{inx} \rangle\right| && (1) \\
=&\left|\langle f_k - f, e^{inx} \rangle\right| && (2) \\
\le&\|f_k - f\|_1 \underset{k\to\infty}{\to} 0 && (3)
\end{aligned}
$$

Remark
Let $\mathrm{Trig}(T)$ denote the set of Trigonometric polynomials on T.
By A3,

$$\overline{\mathrm{Trig}(T)} = L^1(T)$$

Theorem [Riemann-Lebesgue Lemma]

If $f \in L^1(T)$, then $\lim_{|n|\to\infty}\langle f, e^{inx}\rangle = 0$.
Proof:
Let $\epsilon > 0$ be given and let $P \in \text{Trig}(T)$, such that $\|f - P\|_1 < \epsilon$.
Say $P(x) = \sum_{k=-N}^{N} a_k e^{ikx}$.
For $n > N$ or $n < -N$ ($|n| > N$), we have that: $\langle P, e^{inx}\rangle = 0$.
For $|n| > N$,

$$\begin{aligned}\left|\langle f, e^{inx}\rangle\right| &= \left|\langle f - p, e^{inx}\rangle\right| \\ &\leq \|f - P\|_1 < \epsilon.\end{aligned}$$

10.3 Vector-Valued Integration

See PDF

10.4 Summability Kernels

Goal

Given $f \in L^1(T)$, determine when $S_n(f, x) \to f(x)$
Pointwise? In L^1?
Main Tool:

1. Summability Kernels
2. Convolution

Definition:
$f, g \in L^1(T)$.
The convolution of f and g is the function $f * g : T \to \mathbb{C}$ given by

$$\begin{aligned}(f * g)(x) &= \frac{1}{2\pi}\int_T f(t)g(x - t)\, dt \\ &= \frac{1}{2\pi}\int_T f(t)g_t(x)\, dt\end{aligned}$$

Facts

1. Given $f, g \in L^1(T)$, $f * g \in L^1(T)$ as well.
2. $\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$
3. This makes $L^1(T)$ a Banach algebra.

Let $C(T)$ denote the set of continuous functions $T \to \mathbb{C}$.
Definition
A summability kernel is a sequence $(k_n) \subseteq C(T)$ such that:

1. $\frac{1}{2\pi}\int_T k_n = 1$
2. $\exists M, \forall n, \|k_n\|_1 \leq M$.
3. For all $0 < \delta < \pi$:

$$\lim_{n\to\infty}\left(\int_{-\pi}^{-\delta} |k_n| + \int_{\delta}^{\pi} |k_n|\right) = 0$$

Proposition

Let $(B, \|\cdot\|_B)$ be a Banach space. Let $\varphi : T \to B$ be continuous.

Let $(k_n) \subseteq C(T)$ be a summability kernel.

Then,

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_T k_n(t)\varphi(t)\, dt = \varphi(0)$$

in the B-norm.

Proof

Appendix

Notice how (2) and (3) are used.

Remark

By A3, $\varphi : T \to L^1(T)$ given by $\varphi(t) = f_t = f(x-t)$ is continuous.

Theorem

$f \in L^1(T)$, (k_n) is a summability kernel.

In $L^1(T)$,

$$f = \lim_{n\to\infty} k_n * f$$

Proof

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_T k_n(t)\varphi(t)\, dt = \varphi(0)$$

$$\varphi : T \to L^1, t \mapsto f_t$$

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{2\pi} \int_T k_n(t) f(x-t)\, dt = f(x) \\ \Rightarrow &\lim_{n\to\infty} (k_n * f)(x) = f(x) \end{aligned}$$

11 Week 11

11.1 Dirichlet Kernel

Recall

If (k_n) is a summability kernel and $f \in L^1(T)$, then $\lim_{n\to\infty} k_n * f = f$ in $L^1(T)$.

Want

Find (k_n) such that:

$$k_n * f = S_n(f)$$

Remark

$f \in L^1(T)$.

For $n \in \mathbb{Z}$, consider $\varphi_n(x) = e^{inx} \in L^1(T)$.

Then,

$$\begin{aligned}
&\left(\varphi_n * f\right)(x)\\
=&\frac{1}{2\pi}\int_T \varphi_n(t) f_t(x)\,dt\\
=&\frac{1}{2\pi}\int_T e^{int} f(x-t)\,dt\\
=&\frac{1}{2\pi}e^{inx}\int_T e^{-in(x-t)} f(x-t)\,dt\\
\overset{A3}{=}&\frac{1}{2\pi}e^{inx}\int_T e^{-in(-t)} f(-t)\,dt\\
\overset{P}{=}&\frac{1}{2\pi}e^{inx}\int_T e^{-int} f(t)\,dt\\
=&e^{inx}\langle f, e^{inx}\rangle
\end{aligned}$$

<u>Remark</u>
$f \in L^1(T)$. If $P(x) = \sum_{k=-n}^{n} a_k e^{ikx}$
then

$$\begin{aligned}
&\left(P * f\right)(x)\\
=&\frac{1}{2\pi}\int_T P(t) f(x-t)\,dt\\
=&\sum_{k=-n}^{n} \frac{a_n}{2\pi}\int_T e^{ikt} f(x-t)\,dt\\
=&\sum_{k=-n}^{n} a_n \left(\varphi_k * f\right)(x)\\
=&\sum_{k=-n}^{n} a_n e^{ikx}\langle f, e^{ikx}\rangle
\end{aligned}$$

<u>Remark / Definition</u>
Let $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$ be the Dirichlet Kernel of order n.
Thus,

$$\begin{aligned}
&\left(D_n * f\right)(x)\\
=&\sum_{k=-n}^{n} e^{ikx}\langle f, e^{ikx}\rangle\\
=&S_n(f, x) \qquad (n\text{-th partial sum})
\end{aligned}$$

Bad news...
(D_n) is not a summability kernel. (See appendix). But we are close.

11.2 Fejer Kernel

<u>Recall</u>

1. $\lim_{n\to\infty} k_n * f = f$ (in $L^1(T)$)
2. $D_n * f = S_n(f)$
3. D_n is not a summability kernel.

The partial fix...
Idea $(x_n) \subseteq \mathbb{C}$.
Consider

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Exercise:
If $x_n \to x$, then $y_n \to x$.
Definition:

$$F_n(x) = \frac{D_0(x) + D_1(x) + \cdots + D_n(x)}{n+1}$$

Let $F_n(x)$ be the Fejer Kernel of order n.
Remark

$$F_0(x) = D_0(x) = 1$$
$$F_1(x) = \frac{e^{-ix} + 2e^{i0x} + e^{ix}}{2}$$
$$F_2(x) = \frac{e^{-i2x} + 2e^{-ix} + 3e^{i0x} + 2e^{ix} + e^{i2x}}{3}$$
$$\cdots$$
$$F_n(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}$$

Remark
(F_n) is a summability kernel! (See appendix).
Remark / Definition

$$\begin{aligned} F_n * f &= \frac{1}{n+1} \sum_{k=0}^{n} D_k * f \\ &= \frac{1}{n+1} \sum_{k=0}^{n} S_k(f) \\ &= \frac{S_0(f) + S_1(f) + \cdots + S_n(f)}{n+1} \\ &=: \sigma_n(f) \qquad (n\text{th Cesaro mean}) \end{aligned}$$

Theorem
$f \in L^1(T)$, (F_n) being the Fejer kernel.

$$\begin{aligned} &\lim_{n\to\infty} F_n * f \\ &= \lim_{n\to\infty} \sigma_n(f) \\ &= f \qquad \text{in } L^1(T) \end{aligned}$$

Remark:
If $(S_n(f))$ converges in L^1, then

$$S_n(f) \to f$$

in $L^1(T)$.

11.3 Fejer's Theorem

Recall

$\lim_{n\to\infty}\sigma_n(f) = f$ in $L^1(T)$, where $\sigma_n(f) = \frac{S_0(f)+\cdots+S_n(f)}{n+1}$.

Idea

L^1 convergence is great theoretically, but pointwise convergence is practical.

Theroem [Fejer's Theorem]

For $f \in L^1(T)$ and $t \in T$, consider $\omega_f(t) = \frac{1}{2}\lim_{x\to 0^+}(f(t+x)+f(t-x))$ provided the limit exists. Then

$$\sigma_n(f,t) \to \omega_f(t)$$

In particular, if f is continuous at t, then

$$\sigma_n(f,t) \to f(t)$$

Proof: Appendix

In practice:

1. Fix $x \in T$.
2. Prove $(S_n(f,x))$ converges.
3. Then
$$S_n(f,x) \to \omega_f(x)$$
4. If f is continuous at x, then $S_n(f,x) \to f(x)$. i.e., $S(f,x) = f(x)$.

Example:

$f \in L^1(T)$, $f(x) = |x|$.

$$\begin{aligned}
&S_n(f,x)\\
=&a_0 + \sum_{k=1}^{n}(b_k\cos(K_x) + c_k\sin(K_x))
\end{aligned}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{\pi}{2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(kx)\,dx = \frac{2(-1)^k-2}{k^2\pi}$$

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(kx)\,dx = 0$$

Therefore,

$$\begin{aligned}
&S_n(f,x)\\
=&\frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{(-1)^k-1}{k^2}\cos(kx)\right)\\
=&\frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\frac{n+1}{2}}\left(\frac{-2}{(2k-1)^2}\cos((2k-1)x)\right)
\end{aligned}$$

Note: $(S_n(f,x))$ converges by comparison test with $\sum\frac{1}{(2k-1)^2}$.

Since f is continuous,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)^2}$$

1. Taking $x = 0$:

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$
$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

2.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$
$$= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{\pi^2}{8}$$
$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

12 Week 12

12.1 Homogeneous Banach Spaces

Goal

Generalize what we have done for $L^1(T)$ to $L^p(T)$ $(p < \infty)$.
In particular, we look at $L^2(T)$.
Definition:
A homogeneous Banach space is a Banach space $(B, \|\cdot\|_B)$ such that:

1. B is a subspace of $L^1(T)$.
2. $\|\cdot\|_1 \leq \|\cdot\|_B$
3. For all $f \in B$, for all $\alpha \in T$, $f_\alpha \in B$, $\|f_\alpha\|_B = \|f\|_B$.
4. For all $f \in B$, for all $t_0 \in T$,
$$\lim_{t \to t_0} \|f_t - f_{t_0}\|_B = 0.$$

Exercise: $(L^p(T), \|\cdot\|_P)$. $p < \infty$.
Theorem:
Let B be a homogeneous Banach space, (k_n) be the summability kernel.
For all $f \in B$,
$$\lim_{n \to \infty} \|k_n * f - f\|_B = 0$$

Why?

1.
$$\underbrace{\frac{1}{2\pi} \int_T k_n(t) f_t \, dt}_{B-\text{valued}} = \underbrace{k_n * f}_{L^1-\text{valued}}$$

2.
$$\lim_{n \to \infty} \frac{1}{2\pi} \int_T k_n(t) \varphi(t) \, dt = \varphi(0)$$

for all continuous $\varphi : T \to B$.

3. $\varphi : T \to B$, $\varphi(t) = f_t$ is continuous. (For all $f \in B$)

4.
$$\|k_n * f - f\|_B \to 0$$

Remarks

1. Let B be homogeneous Banach space.

   Taking $k_n = F_n$, we have:
$$\|\sigma_n(f) = f\|_B \to 0$$
   for all $f \in B$.

2. Taking $B = L^p(T)$:

   (a) $\|\sigma_n(f) - f\|_p \to 0$

   (b) $\overline{\text{Trig}(T)} = L^p(T)$

Remark:
In $L^2(T)$:

1. $\overline{\text{Trig}(T)} = L^2(T)$

2. $\overline{\text{Span}\{e^{inx} \mid n \in \mathbb{Z}\}} = L^2(T)$

3. $\{e^{inx} \mid n \in \mathbb{Z}\}$ is an orthonormal basis.

4. Let the above orthonormal basis be written as $\{v_1, v_2, v_3, \dots\}$.

   For all $f \in L^2(T)$,
$$\lim_{n\to\infty} \sum_{i=1}^{n} \langle f, v_i \rangle v_i = f.$$

5. If $v = \frac{1}{\sqrt{2\pi}} e^{ikx}$,
$$\begin{aligned}
\langle f, v \rangle v &= \left(\int_T f(x) \frac{1}{\sqrt{2\pi}} e^{-ikx}\, dx \right) \frac{1}{\sqrt{2\pi}} e^{ikx} \\
&= \frac{1}{\sqrt{2\pi}} \left(2\pi \langle f, e^{ikx} \rangle \right) \frac{1}{\sqrt{2\pi}} e^{ikx} \\
&= \langle f, e^{ikx} \rangle e^{ikx}
\end{aligned}$$

6. For all $f \in L^2(T)$,
$$\|S_n(f) - f\|_2 \to 0$$